Comportamiento asintótico de fluidos viscosos con condiciones de deslizamiento sobre fronteras rugosas

Asymptotic behavior of viscous fluids with slip conditions on rugous boundaries

Memoria escrita por

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Introducción

Un problema importante en mecánica de fluidos consiste en la elección adecuada de las condiciones de frontera. Una hipótesis comúnmente aceptada es que si la frontera del dominio es impermeable entonces un fluido viscoso se adhiere completamente a ella. Esta hipótesis se usa habitualmente en diferentes estudios teóricos así como en experimentos numéricos. Suponiendo que $u = u(x)$ es la velocidad del fluido en $x \in \Omega \subset \mathbb{R}^3$, la completa adherencia (o condición de no deslizamiento) se escribe

$$u(x) = 0 \text{ sobre } x \in \partial\Omega. \tag{1}$$

La hipótesis de completa adherencia no fue siempre aceptada en el pasado ya que la mayoría de los efectos de una frontera rugosa en fluidos viscosos no pueden ser descritos en detalle usando esta condición. Por ello, Navier propuso una condición frontera de deslizamiento con fricción. Suponiendo la frontera impermeable está claro que la componente normal de la velocidad debe ser nula. A esta condición se añade la ecuación correspondiente al equlibrio de fuerzas pero escrita solamente para la componente tangencial. Para fluidos gobernados por las ecuaciones de Stokes o de Navier-Stokes la condición de Navier o condición de deslizamiento viene dada por

$$u \cdot \nu = 0, \quad T\left(\frac{\partial u}{\partial \nu} - p\nu + \gamma u\right) = 0 \text{ sobre } \partial\Omega, \tag{2}$$

donde p es la presión, ν el vector normal unitario exterior a $\partial\Omega$, T la proyección ortogonal sobre el espacio tangente a $\partial\Omega$ y $\gamma \geq 0$ es el coeficiente de fricción. Esta condición ofrece más libertad y parece proporcionar soluciones físicamente más aceptables ya que refleja la interacción entre el fluido y la frontera de Ω. Teniendo en cuenta que $p\nu$ es ortogonal al espacio tangente a $\partial\Omega$, la segunda ecuación de (2) es equivalente a

$$T\left(\frac{\partial u}{\partial \nu} + \gamma u\right) = 0, \tag{3}$$

y por tanto la condición de Navier o condición de deslizamiento se puede también escribir como

$$u \cdot \nu = 0, \quad \frac{\partial u}{\partial \nu} + \gamma u \ \text{ proporcional a } \nu \text{ sobre } \partial\Omega. \tag{4}$$

Ha habido diversos intentos en la literatura de proporcionar una rigurosa justificación de la condición de adherencia. Para ello, suponiendo un fluido gobernado por un sistema de Stokes o de Navier-Stokes en un dominio suficientemente rugoso Ω_ε, donde el parámetro ε corresponde a la amplitud de las rugosidades (como una aproximación oscilante del dominio ideal Ω), y verificando la condición de deslizamiento sobre la frontera rugosa Γ_ε, se puede probar que en el límite cuando ε tiende a cero, la solución satisface la condición de adherencia (1). Es decir, se puede probar que las condiciones de deslizamiento sobre una superficie rugosa se transforman asintóticamente en condiciones de adherencia cuando la amplitud de las rugosidades tienden a cero, suponiendo que la energía de las soluciones está uniformemente acotada y que hay suficiente rugosidad en la frontera oscilante. Desde un punto de vista físico, esto justifica matemáticamente que se suela imponer la condición de adherencia para fluidos viscosos.

La anterior afirmación fue probada en [22] para un fluido tridimensional con una frontera descrita por la ecuación

$$x_3 = -\varepsilon\Psi\left(\frac{x_1}{\varepsilon}, \frac{x_2}{\varepsilon}\right) \quad \forall (x_1, x_2) \in \omega, \tag{5}$$

donde ω un conjunto abierto acotado de $\mathbb{R}^2$ y Ψ una función regular periódica tal que

$$\text{Span}\left(\{\nabla\Psi(z') : z' \in \mathbb{R}^2\}\right) = \mathbb{R}^2, \tag{6}$$

o equivalentemente tal que no se verifica $\Psi(z_1, z_2) = \Psi(z_1)$ o $\Psi(z_1, z_2) = \Psi(z_2)$. Generalizaciones de este resultado han sido obtenidos para el caso periódico en [9] y [10]. Además, este tipo de resultados han sido extendidos en [12] a una frontera no periódica

$$x_3 = \Phi_\varepsilon(x_1, x_2) \quad \forall (x_1, x_2) \in \omega, \tag{7}$$

suponiendo que Φ_ε converge $*$-débil a cero en $W^{1,\infty}(\omega)$ y es tal que el soporte de la medida de Young asociada a $\nabla\Phi_\varepsilon$ contiene dos vectores no lineales independientes.

La homogeneización del sistema de Navier-Stokes ha sido estudiada también en [15] para dominios rugosos muy generales, donde en particular no se impone estructura periódica.

Nuestro objetivo en la presente memoria ha sido estudiar la relación entre las condiciones de Navier y de adherencia para rugosidades más débiles que las consideradas en [22]. La descripción por capítulos de nuestros resultados es como sigue:

Capítulo 1.

A lo largo de esta introducción, los puntos x de $\mathbb{R}^3$ se van a descomponer como (x', x_3) con $x' \in \mathbb{R}^2$, $x_3 \in \mathbb{R}$. También usamos la notación x' para denotar un vector genérico de $\mathbb{R}^2$. En este capítulo vamos a considerar una frontera oscilante Γ_ε descrita por

$$\Gamma_\varepsilon = \left\{ x = (x', x_3) \in \omega \times \mathbb{R} : x_3 = -\delta_\varepsilon \Psi \left(\frac{x'}{\varepsilon} \right) \right\}, \quad \Gamma = \omega \times \{0\}, \tag{8}$$

con $\delta_\varepsilon > 0$ un infinitésimo de ε, i.e. $\lim_{\varepsilon \to 0} \delta_\varepsilon / \varepsilon = 0$, $\Psi \in W^{2,\infty}(\mathbb{R}^2)$ no negativa y periódica de periodo el cubo unidad $Z' = (-1/2, 1/2)^2$ y $\omega \subset \mathbb{R}^2$ es un conjunto abierto, conexo, acotado y con frontera lipschitziana.

Tomando

$$\Omega_\varepsilon = \left\{ x = (x', x_3) \in \omega \times \mathbb{R} : -\delta_\varepsilon \Psi \left(\frac{x'}{\varepsilon} \right) < x_3 < 1 \right\}, \quad \Omega = \omega \times (0, 1), \tag{9}$$

estudiamos el comportamiento asintótico de un fluido viscoso gobernado por el sistema de Stokes o de Navier-Stokes en Ω_ε que satisface la condición de Navier sobre la frontera rugosa Γ_ε.

Figure 1: Dominio Ω_ε definido por (9).

Por simplificar, en este resumen nos vamos a limitar a comentar los resultados obtenidos en el caso del sistema de Stokes en Ω_ε imponiendo la condición de Navier sobre Γ_ε y la condición de adherencia en el resto de frontera $\partial\Omega_\varepsilon \setminus \Gamma_\varepsilon$, es decir, nuestro problema se escribe

$$\begin{cases} -\Delta u_\varepsilon + \nabla p_\varepsilon = f \text{ en } \Omega_\varepsilon \\ \text{div } u_\varepsilon = 0 \text{ en } \Omega_\varepsilon \\ u_\varepsilon \cdot \nu = 0 \text{ sobre } \Gamma_\varepsilon, \quad \dfrac{\partial u_\varepsilon}{\partial \nu} + \gamma u_\varepsilon \text{ paralelo a } \nu \text{ sobre } \Gamma_\varepsilon \\ u_\varepsilon = 0 \quad \text{sobre } \partial\Omega_\varepsilon \setminus \Gamma_\varepsilon, \end{cases} \tag{10}$$

con $\gamma \geq 0$, ν el vector normal unitario exterior a Ω_ε en Γ_ε y f una función de $L^2(\Omega_\varepsilon)^3$ (en el Capítulo 1 se consideran segundos miembros más generales).

Se prueba que el sistema (10) posee una única solución $(u_\varepsilon, p_\varepsilon) \in H^1(\Omega_\varepsilon)^3 \times L^2_0(\Omega_\varepsilon)$ (donde $L^2_0(\Omega_\varepsilon)$ denota el espacio de funciones $L^2(\Omega_\varepsilon)$ de integral nula en Ω_ε) y además, existe $C > 0$, que no depende de ε, tal que

$$\|u_\varepsilon\|_{H^1(\Omega_\varepsilon)^3} + \|p_\varepsilon\|_{L^2(\Omega_\varepsilon)} \leq C, \quad \forall \varepsilon > 0. \tag{11}$$

Nuestro objetivo es estudiar el comportamiento asintótico de las sucesiones u_ε y p_ε. Se prueba

Teorema 0.1 *La solución $(u_\varepsilon, p_\varepsilon)$ de (10) satisface*

$$u_\varepsilon \rightharpoonup u \quad en\ H^1(\Omega)^3, \qquad p_\varepsilon \rightharpoonup p \quad en\ L^2(\Omega), \tag{12}$$

donde (u, p) satisface

$$\begin{cases} -\Delta u + \nabla p = f \ \ en\ \Omega \\ div\ u = 0 \ \ en\ \Omega \\ u = 0 \ \ sobre\ \partial\Omega \setminus \Gamma, \\ u_3 = 0 \ \ sobre\ \Gamma = \omega \times \{0\}, \end{cases} \tag{13}$$

que también satisface una condición frontera para la componente tangencial del tensor de esfuerzos que depende del límite

$$\lambda = \lim_{\varepsilon \to 0} \frac{\delta_\varepsilon}{\varepsilon^{\frac{3}{2}}} \in [0, +\infty]. \tag{14}$$

Concretamente se tiene

i) Si $\lambda = 0$, entonces

$$\partial_3 u' + \gamma u' = 0 \quad sobre\ \Gamma. \tag{15}$$

ii) Si $\lambda \in (0, +\infty)$, entonces definiendo $(\widehat{\phi}^i, \widehat{q}^i)$, $i = 1, 2$ como una solución de

$$\begin{cases} -\Delta_z \widehat{\phi}^i + \nabla_z \widehat{q}^i = 0 \quad en\ \mathbb{R}^2 \times (0, +\infty) \\ \mathrm{div}_z\, \widehat{\phi}^i = 0 \quad en\ \mathbb{R}^2 \times (0, +\infty) \\ \widehat{\phi}^i_3(z', 0) + \partial_{z_i} \Psi(z') = 0, \ \partial_{z_3} (\widehat{\phi}^i)'(z', 0) = 0 \\ \widehat{\phi}^i(., z_3), \ \widehat{q}^i(., z_3) \ periódicas\ de\ periodo\ Z' \\ D_z \widehat{\phi}^i \in L^2(Z' \times (0, +\infty))^{3\times 3}, \ \widehat{q}^i \in L^2(Z' \times (0, +\infty)), \end{cases} \tag{16}$$

y $R \in \mathbb{R}^{2\times 2}$ *por*

$$R_{ij} = \int_{\widehat{Q}} D_z\widehat{\phi}^i : D_z\widehat{\phi}^j \, dz, \quad \forall\, i,j \in \{1,2\}, \tag{17}$$

se tiene

$$\partial_3 u' + \gamma u' + \lambda^2 R u' = 0 \quad \textit{sobre } \Gamma. \tag{18}$$

iii) Si $\lambda = +\infty$*, entonces definiendo*

$$W = Span(\{\nabla\Psi(z') : \ z' \in Z'\}), \tag{19}$$

se tiene

$$u' \in W^{\perp} \quad \textit{sobre } \Gamma, \quad \partial_3 u' + \gamma u' \in W. \tag{20}$$

Observación 0.2 *Para* $\lambda = 0$*, el Teorema 0.1 muestra que la rugosidad de* Γ_ε *es muy pequeña y por tanto la solución* $(u_\varepsilon, p_\varepsilon)$ *de* (10) *se comporta como si* Γ_ε *coincidiera con la frontera plana* Γ*. Para* $\lambda \in (0,+\infty)$ *(talla crítica), la condición frontera que satisface el límite* u *de* u_ε *sobre el espacio tangente a* Γ *contiene el nuevo término* $\lambda^2 R u'$*. El efecto de la frontera* Γ_ε *no es despreciable en este caso, haciendo aparecer en el límite este término de fricción. Finalmente, para* $\lambda = +\infty$ *la rugosidad de* Γ_ε *es tal que el límite* u *de* u_ε *no sólo satisface la condición* $u_3 = 0$ *sobre* Γ*, sino que también la velocidad tangencial sobre* Γ*,* u'*, es ortogonal a los vectores* $\nabla\Psi(z')$*, con* $z' \in Z'$*. En particular, si el espacio* W *definido por* (19) *tiene dimensión 2, entonces* u *satisface la condición de adherencia* $u = 0$ *sobre* Γ*. Esto extiende al caso donde*

$$\lim_{\varepsilon\to 0} \frac{\delta_\varepsilon}{\varepsilon} = 0, \quad \lim_{\varepsilon\to 0} \frac{\delta_\varepsilon}{\varepsilon^{\frac{3}{2}}} = +\infty,$$

los resultados obtenidos en [22] para $\delta_\varepsilon = \varepsilon$*.*

Observación 0.3 *El caso* $\lambda \in (0,+\infty)$ *se puede considerar como el general, tomando* λ *que tiende a cero o a infinito en* (18) *se obtiene* (15) *y* (20) *respectivamente.*

Observación 0.4 *La demostración del Teorema 0.1 se basa en el método "unfolding", [5], [20], [27], que está muy relacionado con el método de convergencia en dos-escalas, [1], [41], [43].*

Se puede además probar el siguiente resultado de convergencia fuerte (resultado de corrector)

Teorema 0.5 *En las condiciones del Teorema 0.1, se tiene* *i) Si $\lambda = 0$ o $+\infty$, entonces*

$$\lim_{\varepsilon\to 0}\Big(\|u_\varepsilon\|_{H^1(\Omega_\varepsilon\setminus\Omega)^3}+\|p_\varepsilon\|_{L^2(\Omega_\varepsilon\setminus\Omega)}+\|u_\varepsilon-u\|_{H^1(\Omega)^3}+\|p_\varepsilon-p\|_{L^2(\Omega)}\Big)=0. \qquad (21)$$

ii) Si $\lambda\in(0,+\infty)$, entonces, tomando $(\widehat{\phi}^i,\widehat{q}^i)$, $i=1,2$ como una solución de (16) y definiendo $\breve{u}_\varepsilon$ y $\breve{p}_\varepsilon$ por

$$\left\{\begin{array}{l} \breve{u}_\varepsilon(x)=u(x)+\lambda\sqrt{\varepsilon}\Big(u_1(x',0)\widehat{\phi}^1(\dfrac{x}{\varepsilon})+u_2(x',0)\widehat{\phi}^2(\dfrac{x}{\varepsilon})\Big),\\ \breve{p}_\varepsilon(x)=p(x)+\dfrac{\lambda}{\sqrt{\varepsilon}}\Big(u_1(x',0)\widehat{q}^1(\dfrac{x}{\varepsilon})+u_2(x',0)\widehat{q}^2(\dfrac{x}{\varepsilon})\Big), \end{array}\right. \qquad (22)$$

se tiene (suponiendo u suficientemente regular)

$$\lim_{\varepsilon\to 0}\Big(\|u_\varepsilon\|_{H^1(\Omega_\varepsilon\setminus\Omega)^3}+\|p_\varepsilon\|_{L^2(\Omega_\varepsilon\setminus\Omega)}+\|u_\varepsilon-\breve{u}_\varepsilon\|_{H^1(\Omega)^3}+\|p_\varepsilon-\breve{p}_\varepsilon\|_{L^2(\Omega)}\Big)=0. \qquad (23)$$

Observación 0.6 *Los resultados obtenidos en este capítulo están publicados en [25].*

Capítulo 2.

Nuestro objetivo en este capítulo es mejorar el resultado de corrector probado en el Teorema 0.5 obteniendo una estimación entre la solución $(u_\varepsilon, p_\varepsilon)$ de (10) y su corrector. Nos hemos centrado en el caso $\lambda\in(0,+\infty)$ que, como ya hemos comentado (in Remark 0.3, puede considerarse como el caso general y donde el problema es más complejo debido a la aparición de términos frontera en el corrector. En este capítulo suponemos que Ω_ε, Γ_ε están dados por

$$\Omega_\varepsilon=\left\{x=(x',x_3)\in\omega\times\mathbb{R}:\ -\lambda\varepsilon^{\frac{3}{2}}\Psi(\frac{x'}{\varepsilon})<x_3<1\right\},$$

$$\Gamma_\varepsilon=\left\{x=(x',x_3)\in\omega\times\mathbb{R}:\ x_3=-\lambda\varepsilon^{\frac{3}{2}}\Psi(\frac{x'}{\varepsilon})\right\},$$

con $\lambda\in(0,+\infty)$.

Nuestro principal resultado es el siguiente Teorema.

Teorema 0.7 *Suponemos que la función u definida por (13)-(18) pertenece a $H^s(\Omega)^3$, con $s>3/2$. Entonces, la solución $(u_\varepsilon,p_\varepsilon)$ de (10) y las funciones $\breve{u}_\varepsilon$, $\breve{p}_\varepsilon$ definidas en (22) satisfacen*

$$\|u_\varepsilon\|_{H^1(\Omega_\varepsilon\setminus\Omega)^3}+\|p_\varepsilon\|_{L^2(\Omega_\varepsilon\setminus\Omega)}+\|u_\varepsilon-\breve{u}_\varepsilon\|_{H^1(\Omega)^3}+\|p_\varepsilon-\breve{p}_\varepsilon\|_{L^2(\Omega)}\le C\sqrt{\varepsilon}. \qquad (24)$$

Para probar el Teorema 0.7, a la hora de estimar la diferencias $u_\varepsilon - \breve{u}_\varepsilon$ y $p_\varepsilon - \breve{p}_\varepsilon$, nos encontramos con el problema de que $\breve{u}_\varepsilon$ y $\breve{p}_\varepsilon$ solo están definidas en Ω y no en Ω_ε. Para salvar esta dificultad aplicamos el cambio de variables $t = \eta_\varepsilon(x)$, con $\eta_\varepsilon : \Omega_\varepsilon \to \Omega$ dado por

$$\eta_\varepsilon(x) = \left(x', \frac{x_3 + \lambda \varepsilon^{\frac{3}{2}} \Psi(\frac{x'}{\varepsilon})}{1 + \lambda \varepsilon^{\frac{3}{2}} \Psi(\frac{x'}{\varepsilon})} \right), \quad \forall\, x \in \Omega_\varepsilon,$$

a las funciones $\breve{u}_\varepsilon$, $\breve{p}_\varepsilon$, obteniendo las funciones

$$u_\varepsilon^*(x) = \breve{u}_\varepsilon(\eta_\varepsilon(x)), \quad p_\varepsilon^*(x) = \breve{p}_\varepsilon(\eta_\varepsilon(x)), \tag{25}$$

que tienen la ventaja de estar definidas en Ω_ε. Usando estas funciones probamos la siguiente estimación,

$$\|u_\varepsilon - u_\varepsilon^*\|_{H^1(\Omega_\varepsilon)^3} + \|p_\varepsilon - p_\varepsilon^*\|_{L^2(\Omega_\varepsilon)} \leq C\sqrt{\varepsilon}, \tag{26}$$

que implica (24).

Capítulos 3 y 4.

A partir de ahora, por coherencia con la notación usual en dominios delgados, vamos a considerar ε como la altura del dominio, lo que implica que el periodo de las rugosidades se notará por r_ε. El objetivo de los Capítulos 3 y 4 es extender los resultados obtenidos en el Capítulo 1 al caso de un fluido viscoso en un dominio delgado de altura ε. Concretamente, el Capítulo 3 corresponde a una publicación en una Nota CRAS ([26]) donde se considera la ecuación de Stokes y donde los resultados se presentan sin demostración. En el Capítulo 4 consideramos la ecuación de Navier-Stokes, aquí las demostraciones sí serán detalladas.

Similarmente a lo que hicimos en el caso de un dominio de "altura fija" nos limitaremos, a fin de simplificar la exposición en esta introducción, al caso de la ecuación de Stokes.

Definimos el dominio rugoso $\Omega_\varepsilon^{thin}$ y la frontera rugosa $\Gamma_\varepsilon^{thin}$ por

$$\Omega_\varepsilon^{thin} = \left\{ x = (x', x_3) \in \omega \times \mathbb{R} : \, -\delta_\varepsilon \Psi\left(\frac{x'}{r_\varepsilon}\right) < x_3 < \varepsilon \right\} \tag{27}$$

$$\Gamma_\varepsilon^{thin} = \left\{ x = (x', x_3) \in \omega \times \mathbb{R} : \, x_3 = -\delta_\varepsilon \Psi\left(\frac{x'}{r_\varepsilon}\right) \right\}, \tag{28}$$

donde los parámetros r_ε, δ_ε son positivos y satisfacen las siguientes relaciones

$$\lim_{\varepsilon \to 0} \frac{r_\varepsilon}{\varepsilon} = 0, \quad \lim_{\varepsilon \to 0} \frac{\delta_\varepsilon}{r_\varepsilon} = 0.$$

Esta geometría describe un dominio delgado de altura ε (en la dirección x_3) y de una capa oscilante de altura $\|\Psi\|_{L^\infty}\delta_\varepsilon$ (observar que $\delta_\varepsilon << \varepsilon$) con oscilaciones de periodo r_ε, que es mucho más pequeño que ε y mucho más grande que δ_ε.

Consideramos $f = (f', f_3)$ que solo depende de las componentes x', $f = f(x')$, es decir, no va a depender de la altura. En realidad, puesto que el grosor del fluido es delgado, para una función suficientemente regular que depende de (x', x_3) se entiende que $f(x', x_3) \sim f(x', 0)$. También se pueden considerar funciones de la forma $f(x', x_3/\varepsilon)$.

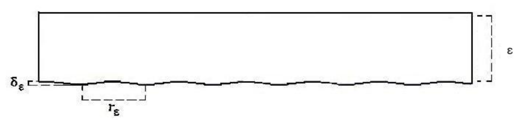

Figure 2: Dominio delgado $\Omega_\varepsilon^{thin}$ definido por (27)

Así, para $f = (f', f_3) \in L^2(\omega)^3$, similarmente a (10), consideramos el problema de Stokes

$$\begin{cases} -\Delta u_\varepsilon + \nabla p_\varepsilon = f \quad \text{en } \Omega_\varepsilon^{thin} \\ \operatorname{div} u_\varepsilon = 0 \quad \text{en } \Omega_\varepsilon^{thin} \\ u_\varepsilon \cdot \nu = 0 \quad \text{sobre } \Gamma_\varepsilon^{thin} \qquad \dfrac{\partial u_\varepsilon}{\partial \nu} + \gamma u_\varepsilon \quad \text{paralelo a } \nu \quad \text{sobre } \Gamma_\varepsilon^{thin} \\ u_\varepsilon = 0 \quad \text{sobre } \partial\Omega_\varepsilon^{thin} \setminus \Gamma_\varepsilon^{thin}. \end{cases} \tag{29}$$

Probamos que (29) posee una única solución $(u_\varepsilon, p_\varepsilon) \in H^1(\Omega_\varepsilon^{thin})^3 \times L_0^2(\Omega_\varepsilon^{thin})$. Además, se tienen las siguientes estimaciones

$$⨍_{\Omega_\varepsilon^{thin}} |u_\varepsilon|^2 dx \le C\varepsilon^4, \qquad ⨍_{\Omega_\varepsilon^{thin}} |Du_\varepsilon|^2 dx \le C\varepsilon^2, \qquad ⨍_{\Omega_\varepsilon^{thin}} |p_\varepsilon|^2 dx \le C. \tag{30}$$

El objetivo es estudiar el comportamiento asintótico de u_ε y p_ε cuando ε tiende a cero. Para este propósito, como es usual, usamos una dilatación en la variable x_3 para tener las funciones definidas en una conjunto abierto de altura fija. Es decir, definimos $\tilde{u}_\varepsilon \in H^1(\Omega)^3$, $\tilde{p}_\varepsilon \in L_0^2(\Omega)$ por

$$\tilde{u}_\varepsilon(y) = u_\varepsilon(y', \varepsilon y_3), \quad \tilde{p}_\varepsilon(y) = p_\varepsilon(y', \varepsilon y_3), \quad \text{e.c.t. } y \in \Omega, \tag{31}$$

y probamos

Teorema 0.8 *Sea* $(u_\varepsilon, p_\varepsilon) \in H^1(\Omega_\varepsilon^{thin})^3 \times L_0^2(\Omega_\varepsilon^{thin})$ *la solución del sistema de Stokes* (29) *y sean* $\tilde{u}_\varepsilon$, $\tilde{p}_\varepsilon$ *definidas por (31). Entonces, existen* $v' \in H^1(0,1;L^2(\omega))^2$, $w \in H^2(0,1;H^{-1}(\omega))$ *y* $p \in L_0^2(\Omega)$, *p independiente de la componente en altura* y_3, *tales que*

$$\frac{\tilde{u}_\varepsilon}{\varepsilon} \rightharpoonup 0 \ \text{en} \ H^1(\Omega)^3, \ \frac{\tilde{u}_\varepsilon}{\varepsilon^2} \rightharpoonup (v', 0) \ \text{en} \ H^1(0,1;L^2(\omega))^3, \ \frac{\tilde{u}_{\varepsilon,3}}{\varepsilon^3} \rightharpoonup w \ \text{en} \ H^2(0,1;H^{-1}(\omega)) \quad (32)$$

$$\tilde{p}_\varepsilon \rightharpoonup p \ \text{en} \ L^2(\Omega), \quad (33)$$

donde, dependiendo de los valores de λ_{thin} *definido por*

$$\lambda_{thin} = \lim_{\varepsilon \to 0} \frac{\delta_\varepsilon}{r_\varepsilon^{3/2}} \varepsilon^{1/2} \in [0, +\infty], \quad (34)$$

las funciones v', w *y* p *están dadas por*

(i) Si $\lambda_{thin} = 0$, *la función p es solución de*

$$\begin{cases} -\text{div}_{y'} \left(\left(\frac{1}{3} + (1+\gamma)^{-1} \right) (\nabla_{y'} p - f') \right) = 0 \ \text{en} \ \omega, \\ \left(\left(\frac{1}{3} + (1+\gamma)^{-1} \right) (\nabla_{y'} p - f') \right) \cdot \nu = 0 \ \ \text{sobre} \ \partial\omega, \end{cases} \quad (35)$$

la función v' *viene dada por*

$$v'(y) = \frac{1}{2} \left(y_3^2 + (1+\gamma)^{-1} \right) (\nabla_{y'} p(y') - f'(y')), \qquad p.c.t. \ y \in \Omega, \quad (36)$$

y la distribución $w = 0$.

(ii) Si $\lambda_{thin} \in (0, +\infty)$, *entonces definiendo R por* (17), *se tiene que la función p es la solución de la ecuación de Reynolds*

$$\begin{cases} -\text{div}_{y'} \left(\left(\frac{1}{3} I + \left((1+\gamma) I + \lambda_{thin}^2 R \right)^{-1} \right) (\nabla_{y'} p - f') \right) = 0 \ \ \text{en} \ \omega, \\ \left(\left(\frac{1}{3} I + \left((1+\gamma) I + \lambda_{thin}^2 R \right)^{-1} \right) (\nabla_{y'} p - f') \right) \cdot \nu = 0 \ \ \text{sobre} \ \partial\omega, \end{cases} \quad (37)$$

la función v' *está dada por*

$$v'(y) = \frac{(y_3 - 1)}{2} \left(y_3 I + \left((1+\gamma) I + \lambda_{thin}^2 R \right)^{-1} \right) (\nabla_{y'} p(y') - f'(y')), \qquad p.c.t. \ y \in \Omega, \quad (38)$$

y la distribución w

$$w(y) = -\int_0^{y_3} div_{y'} v(y', s)\, ds \ \ \text{en} \ \Omega. \quad (39)$$

(iii) Si $\lambda_{thin} = +\infty$, entonces denotando por $P_{W^\perp}$ la proyección ortogonal de $\mathbb{R}^2$ sobre el ortogonal de W definido por (19), *se tiene que la función p es la solución del problema de Reynolds*

$$\begin{cases} -div_{y'}\left(\left(\frac{1}{3}I + (1+\gamma)^{-1}P_{W^\perp}\right)(\nabla_{y'}p - f')\right) = 0 \quad en\ \omega \\ \left(\left(\frac{1}{3}I + (1+\gamma)^{-1}P_{W^\perp}\right)(\nabla_{y'}p - f')\right)\cdot \nu = 0 \quad sobre\ \partial\omega. \end{cases} \tag{40}$$

La función v' está dada por

$$v'(y) = \frac{(y_3-1)}{2}\left(y_3 I + (1+\gamma)^{-1}P_{W^\perp}\right)(\nabla_{y'}p(y') - f'(y')), \qquad e.c.t.\ y \in \Omega \tag{41}$$

y la distribución w viene dada por (39).

Observación 0.9 *El parámetro λ_{thin} en el Teorema 0.8 juega un papel similar al de λ definido en el Teorema 0.1, es decir si $\lambda_{thin} = 0$, se tiene que la rugosidad es demasiado suave y no tiene efecto en la solución, $\lambda_{thin} = \infty$, la rugosidad es tan fuerte que hace que en Γ, v' pertenezca al ortogonal del espacio W definido por (19). El caso $\lambda_{thin} \in (0, +\infty)$ es el caso crítico donde la rugosidad no es tan fuerte como para implicar una condición de adherencia en el límite pero si lo suficientemente para hacer aparecer un término de fricción. Observamos que tomando $\varepsilon = 1$ en (34), los parámetros λ definido por* (14) *y λ_{thin} definido por* (34) *coinciden (porque en* (34) *r_ε denota en realidad el tamaño del periodo que se denotaba por ε en* (14)*). En el caso de dominios delgados que estamos estudiando en este capítulo, la expresión de λ_{thin} depende no sólo de los parámetros δ_ε, r_ε que definen Γ_ε sino también de la altura ε de Ω_ε. Esto se debe al hecho de que lejos de la frontera rugosa el comportamiento del fluido es diferente al comportamiento del fluido estudiado en el Capítulo 1.*

Observación 0.10 *La prueba del Teorema 0.8 se basa en la idea usada en el Capítulo 1, es decir el "unfolding method", que se usa para estudiar el comportamiento del fluido cerca de Γ_ε junto con el cambio de variables* (31) *que se usa para estudiar su comportamiento lejos de Γ_ε.*

Finalmente, en el siguiente teorema damos resultados de corrector para la velocidad y la presión.

Teorema 0.11 *Sean $(u_\varepsilon, p_\varepsilon) \in H^1(\Omega_\varepsilon^{thin})^3 \times L_0^2(\Omega_\varepsilon^{thin})$ la solución del sistema de Stokes* (29) *y $(\tilde{u}_\varepsilon, \tilde{p}_\varepsilon)$ definidas por (31). Suponemos también que existe el límite λ_{thin} dado por* (34)*. Entonces,* *i) Si* $\lambda_{thin} = 0$ *o* $+\infty$*, definiendo* $\breve{u}_\varepsilon^{thin}$*,* $\breve{p}_\varepsilon^{thin}$ *por*

$$\breve{u}_\varepsilon^{thin}(x) = \Big(\varepsilon^2 v'(x', \frac{x_3}{\varepsilon}), 0\Big), \quad \breve{p}_\varepsilon^{thin}(x) = p(x') \quad e.c.t.\ x \in \Omega_\varepsilon^{thin}, \tag{42}$$

se tiene

$$\lim_{\varepsilon\to 0} \frac{1}{\varepsilon^5}\left(\int_{\Omega_\varepsilon\setminus(\omega\times(0,\varepsilon))} |u_\varepsilon|^2 dx + \int_{\omega\times(0,\varepsilon)} |u_\varepsilon - \breve{u}_\varepsilon^{thin}|^2 dx\right) = 0 \tag{43}$$

$$\lim_{\varepsilon\to 0} \frac{1}{\varepsilon^3}\left(\int_{\Omega_\varepsilon\setminus(\omega\times(0,\varepsilon))} |D(u_\varepsilon - \breve{u}_\varepsilon^{thin})|^2 dx + \int_{\omega\times(0,\varepsilon)} |D(u_\varepsilon - \breve{u}_\varepsilon^{thin})|^2 dx\right) = 0, \tag{44}$$

$$\lim_{\varepsilon\to 0} \frac{1}{\varepsilon}\left(\int_{\Omega_\varepsilon\setminus(\omega\times(0,\varepsilon))} |p_\varepsilon|^2 dx + \int_{\omega\times(0,\varepsilon)} |p_\varepsilon - \breve{p}_\varepsilon^{thin}|^2 dx\right) = 0. \tag{45}$$

ii) Si $\lambda_{thin} \in (0, +\infty)$*, las afirmaciones* (43)-(44)-(45) *siguen verificándose reemplazando* $\breve{u}_\varepsilon^{thin}$ *por*

$$\breve{u}_\varepsilon^{thin}(x) = \Big(\varepsilon^2 v'(x', \frac{x_3}{\varepsilon}), 0\Big) + \lambda_{thin}\varepsilon^{3/2} r_\varepsilon^{1/2}\left(v_1'(x',0)\widehat{\phi}^1(\frac{x}{r_\varepsilon}) + v_2'(x',0)\widehat{\phi}^2(\frac{x}{r_\varepsilon})\right).$$

Capítulo 5.

En los capítulos anteriores nos hemos ocupado de fronteras rugosas periódicas. El presente capítulo está dedicado al estudio de fronteras más generales. Nuestros resultados se encuentran relacionados con los obtenidos en [15] donde se obtiene el problema límite del problema (10) para una sucesión de dominios Ω_ε bastante general. En nuestro caso nos centraremos en sistemas elípticos, especialmente en el sistema de la elasticidad lineal, con condiciones de contorno más generales que las consideradas hasta ahora.

Consideramos una sucesión de conjuntos abiertos Lipschitz $\Omega_n \subset \mathbb{R}^N$, que convergen a un conjunto abierto Lipschitz $\Omega \subset \mathbb{R}$ en el siguiente sentido: Para todo $\rho > 0$, existe $n_0 \in \mathbb{N}$ tal que para todo $n \geq n_0$,

$$\Omega^{\rho^-} = \{x \in \Omega : d(x, \partial\Omega) < \rho\} \subset \Omega_n \subset \{x \in \mathbb{R}^N : d(x, \overline{\Omega}) < \rho\} = \Omega^{\rho^+}. \tag{46}$$

También consideramos un tensor de cuarto orden A con coeficientes en $L^\infty(\widetilde{\Omega})$, donde $\widetilde{\Omega}$ es un conjunto abierto regular tal que $\overline{\Omega} \subset \widetilde{\Omega} \subset \mathbb{R}^N$, es decir $A \in L^\infty(\widetilde{\Omega})^{M\times N\times M\times N}$, de manera que

$$\int_{\widetilde{\Omega}} ADu : Dv\, dx = \int_{\widetilde{\Omega}} \sum_{i,j=1}^{N} \sum_{\alpha,\beta=1}^{M} A_{i\alpha j\beta} \frac{\partial u_\alpha}{\partial x_i} \frac{\partial v_\beta}{\partial x_j}\, dx,$$

esté correctamente definido para toda u, v en $H^1(\widetilde{\Omega})^M$, y tal que existe $\alpha > 0$ satisfaciendo

$$\alpha\|v\|^2_{H^1(\Omega_n)^M} \leq \int_{\Omega_n} ADv : Dv\,dx, \quad \forall\, v \in H^1(\Omega_n)^M, \text{ con } v(x) \in V_n(x),$$

donde para cada $x \in \overline{\Omega}_n$, denotamos por $V_n(x)$ un cierto subespacio vectorial de $\mathbb{R}^M$. Observar que en el caso $N = M$, si para $x \in \partial\Omega_n$ tomamos como $V_n(x)$ el espacio tangente a $\partial\Omega_n$ en x, la condición $v(x) \in V_n(x)$ implica la condición de impermeabilidad $v\nu = 0$ sobre $\partial\Omega_n$. Al tomar en este capítulo subespacios vectoriales V_n arbitrarios estamos trabajando en un marco que recoge una gran diversidad de problemas de homogeneización.

Entonces, consideramos el problema de homogeneización

$$\begin{cases} u_n \in V_n \ \text{ e.q.t. } \overline{\Omega}_n \\ \displaystyle\int_{\Omega_n} ADu_n : Dv\,dx = \int_{\Omega_n} f_n \cdot v\,dx + \int_{\Omega_n} G_n : Dv\,dx, \ \forall\, v \in H^1(\Omega_n)^M, \ v \in V_n \text{ e.q.t. } \overline{\Omega}_n, \end{cases} \tag{47}$$

donde f_n y G_n son sucesiones acotadas en $L^2(\Omega_n)^M$ y $L^2(\Omega_n)^{M\times N}$ respectivamente, que convergen a algún $f \in L^2(\Omega)^M$ y $G \in L^2(\Omega)^{M\times N}$ en el siguiente sentido

$$f_n \rightharpoonup f \text{ in } L^2(\Omega^{\rho^-})^M, \quad G_n \to G \text{ en } L^2(\Omega^{\rho^-})^{M\times N}, \quad \forall\, \rho > 0. \tag{48}$$

y G_n tal que

$$\lim_{\rho\to 0} \limsup_{n\to\infty} \int_{\Omega_n \setminus \Omega^{\rho^-}} |G_n|^2 dx = 0. \tag{49}$$

El principal resultado de este capítulo es el siguiente teorema

Teorema 0.12 *Existen una subsucesión de n, que continuaremos denotando por n, una medida de Radon μ en $\overline{\Omega}$ que se anula en conjuntos de capacidad nula, una función μ-medible $R : \overline{\Omega} \to \mathcal{M}_{N\times N}$ tal que*

$$R\xi \cdot \xi \geq 0, \ |R\xi \cdot \eta| \leq \beta (R\xi \cdot \xi)^{\frac{1}{2}} (R\eta \cdot \eta)^{\frac{1}{2}}, \quad \forall\, \xi, \eta \in \mathbb{R}^N, \ \mu\text{-e.c.t. } \overline{\Omega},$$

para algún $\beta > 0$ y una aplicación V de $\overline{\Omega}$ en el conjunto de subespacios lineales en $\mathbb{R}^N$, satisfaciendo

$$\begin{gathered} \alpha\|v\|^2_{H^1(\Omega)^M} \leq \int_\Omega ADv : Dv\,dx + \int_{\overline{\Omega}} Ru \cdot u\,d\mu \\ \forall\, v \in H^1(\Omega)^M, \ v(x) \in V(x) \ \text{e.q.t. } x \in \overline{\Omega}, \end{gathered} \tag{50}$$

tal que para todo $\rho > 0$ las soluciones de (47) *convergen débilmente en* $H^1(\Omega^{\rho^-})^M$ *a la solución única* u *del problema variacional*

$$\begin{cases} u \in H^1(\Omega)^M, \ u \in V \ \textit{e.q.t.} \ \overline{\Omega}, \ \displaystyle\int_{\overline{\Omega}} Ru \cdot u \, d\mu < +\infty \\ \displaystyle\int_\Omega ADu : Dv \, dx + \int_\Omega Ru \cdot v \, d\mu = \int_\Omega f \cdot v \, dx + \int_\Omega G : Dv \, dx \\ \forall \, v \in H^1(\Omega)^M, \ v \in V \ \textit{e.q.t.} \ \overline{\Omega}, \ \displaystyle\int_{\overline{\Omega}} Rv \cdot v \, d\mu < +\infty. \end{cases} \tag{51}$$

La subsucesión de n*, la medida* μ *y las aplicaciones* R *y* V *no dependen de* f_n*,* G_n*,* f *y* G*.*

Observación 0.13 *Más generalmente mostraremos que el problema* (51) *es estable por homogeneización.*

Suponiendo que existe un subconjunto cerrado C_n *tal que* $V_n = \{0\}$ *en* C_n*,* $V_n = \mathbb{R}^N$ *en* $\Omega_n \setminus C_n$ *y* V_n *arbitrario en* $\partial\Omega_n \setminus C_n$*, el problema* (47) *puede escribirse como*

$$\begin{cases} -\mathrm{div}(ADu_n - G_n) = f_n \quad \textit{en } \Omega_n \setminus C_n \\ u_n = 0 \ \textit{sobre } C_n, \\ u_n \in V_n, \ (ADu_n - G_n) \cdot \nu \in V_n^\perp \quad \textit{sobre } \partial\Omega_n \setminus C_n \end{cases} \tag{52}$$

con ν *el vector normal unitario exterior a* Ω_n *sobre* $\partial\Omega_n$*. Observar que en el caso* $\Omega_n = \Omega$ *y* $V_n = \{0\}$ *sobre* $\partial\Omega_n$*, el problema* (52) *es el clásico problema de homogeneización para ecuaciones elípticas lineales en dominios que varían con condiciones tipo Dirichlet. En este caso, el término* $Ru\mu$ *que aparece en la ecuación límite es lo que, en la terminología de D. Cioranescu y F. Murat, es conocido como el término extraño (ver por ejemplo [18], [19], [23], [30], [32], [34], [35], [36], [37], para la homogeneización de problemas elípticos lineales y no-lineales en dominios variables con condiciones tipo Dirichlet). Tomando* $C_n = \emptyset$*, el problema* (52) *permite incorporar distintas condiciones frontera. En este caso es simple comprobar que en* (51)*, la medida* μ *está concentrada en* $\partial\Omega$ *y que* $V = \mathbb{R}^N$ *en* Ω*. Por lo tanto* (51) *se puede escribir (al menos formalmente) como el siguiente problema con una condición de Fourier generalizada*

$$\begin{cases} -div(ADu - G) = f \quad \textit{en } \Omega \\ \displaystyle\int_{\partial\Omega} Ru \cdot u \, d\mu < +\infty, \quad u \in V \ \textit{sobre } \partial\Omega, \quad (ADu - B)\nu + Ru\mu \in V^\perp \ \ \textit{sobre } \partial\Omega. \end{cases}$$

En particular, para $V_n = \mathbb{R}^N$ *en* Ω_n *y* V_n *tomando solamente los valores* $\{0\}$ *y* $\mathbb{R}^N$ *sobre la frontera, el problema* (52) *corresponde a la homogeneización de un problema elíptico en* Ω_n

donde imponemos una condición tipo Dirichlet sobre un subconjunto de la frontera variable y una condición de tipo Neumann sobre el resto de la frontera. Este problema ha sido estudiado en [16], [17]. Una diferencia entre este trabajo y las referencias mencionadas para la homogeneización de problemas de Dirichlet es que la hipótesis de elipticidad (50) *impuesta a los operadores está escrita en forma integral en lugar de forma puntual. Esto es más conveniente en particular para el sistema elasticidad lineal, donde el tensor solo depende de la parte simétrica de la derivada, caso que será estudiado como ejemplo en el Capítulo 5, mostrando como nuestros resultados se aplican aquí suponiendo*

$$\Omega_\varepsilon = \left\{(x', x_N) \in \mathbb{R}^N : \ x' \in \omega, \ 0 \leq x_N \leq \psi_n(x')\right\}$$

con ω Lipschitz y ψ_n una sucesión acotada en $W^{1,\infty}(\omega)$.

Como hemos dicho anteriormente un resultado similar al del Teorema 0.12, para la homogeneización del sistema de Navier-Stokes en dominios rugosos satisfaciendo la condicion de deslizamiento sobre la frontera, ha sido obtenido en [15]. Los resultados en [15] se basan en una teorema de representación integral que aparece en [33] y que se encuentra adaptado al uso de técnicas de Γ-convergencia. Análogamente nuestro resultado está basado en una variante de este teorema de representación más adaptado a técnicas de H-convergencia. El resultado que aparece en [33] es válido para funcionales convexos y permite por tanto trabajar con EDPs no lineales. Nuestra variante se refiere a funcionales cuadráticos y por lo tanto sólo es válido para EDPs lineales, pero tiene la ventaja de que no supone que el funcional sea convexo y así, el término de difusión de la EDP no tiene que ser necesariamente simétrico.

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Chapter 1

Asymptotic behavior of a viscous fluid with slip boundary conditions on a slightly rough wall

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Abstract.

For an oscillating boundary of period and amplitude ε, it is known that the asymptotic behavior when ε tends to zero of a three-dimensional viscous fluid satisfying slip boundary conditions is the same as if we assume no-slip (adherence) boundary conditions. In the present paper we consider the case where the period is still ε but the amplitude is δ_ε with $\delta_\varepsilon/\varepsilon$ converging to zero. We show that if $\delta_\varepsilon/\varepsilon^{\frac{3}{2}}$ tends to infinity, the equivalence between the slip and no-slip conditions still holds. If the limit of $\delta_\varepsilon/\varepsilon^{\frac{3}{2}}$ belongs to $(0,+\infty)$ (critical size) then we still have the slip boundary conditions in the limit but with a bigger friction coefficient. In the case where $\delta_\varepsilon/\varepsilon^{\frac{3}{2}}$ tends to zero the boundary behaves as a plane boundary. Besides the limit equation, we also obtain an approximation (corrector result) of the pressure and the velocity in the strong topology of L^2 and H^1 respectively.

1.1 Introduction

For a viscous fluid in a three-dimensional domain such that its boundary is covered by microscopic periodic asperities, it has been proved that to impose slip or no-slip conditions on the boundary is asymptotically equivalent. From a physical point of view, this justifies that no-slip (adherence) conditions on the boundary are usually imposed for viscous fluids. The above assertion was proved in [12] for a boundary described by the equation

$$x_3 = -\varepsilon\Psi(\frac{x_1}{\varepsilon}, \frac{x_2}{\varepsilon}) \quad \forall (x_1, x_2) \in \omega, \tag{1.1}$$

with $\varepsilon > 0$ devoted to converge to zero, ω a bounded open set of $\mathbb{R}^2$ and Ψ a smooth periodic function such that

$$Span(\{\nabla\Psi(y') : \ y' \in \mathbb{R}^2\}) = \mathbb{R}^2. \tag{1.2}$$

These results have been generalized in [6] to a non-periodic boundary

$$x_3 = \Phi_\varepsilon(x_1, x_2) \quad \forall (x_1, x_2) \in \omega, \tag{1.3}$$

assuming that Φ_ε converges $*$-weakly to zero in $W^{1,\infty}(\omega)$ and it is such that the support of the Young's measure associated to $\nabla\Phi_\varepsilon$ contains two non-linear independent vectors.

In the present paper, we are interested in the case of a weak rugosity described by

$$\Gamma_\varepsilon = \left\{ x = (x_1, x_2, x_3) \in \omega \times \mathbb{R} : \ x_3 = -\delta_\varepsilon\Psi(\frac{x_1}{\varepsilon}, \frac{x_2}{\varepsilon}) \right\}, \tag{1.4}$$

where $\delta_\varepsilon > 0$ satisfies

$$\lim_{\varepsilon\to 0} \frac{\delta_\varepsilon}{\varepsilon} = 0,$$

and where Ψ is periodic and smooth. Remark that the assumptions imposed in [6] are not satisfied in this case. Indeed, if $\Phi_\varepsilon(x_1, x_2) = -\delta_\varepsilon\Psi(\frac{x_1}{\varepsilon}, \frac{x_2}{\varepsilon})$, then $\nabla\Phi_\varepsilon(x_1, x_2) = -\frac{\delta_\varepsilon}{\varepsilon}\nabla\Psi(\frac{x_1}{\varepsilon}, \frac{x_2}{\varepsilon})$ converges strongly to zero in $L^\infty(\omega)^2$.

Taking

$$\Omega_\varepsilon = \left\{ x = (x_1, x_2, x_3) \in \omega \times \mathbb{R} : \ -\delta_\varepsilon\Psi\left(\frac{x_1}{\varepsilon}, \frac{x_2}{\varepsilon}\right) < x_3 < 1 \right\}, \quad \Omega = \omega \times (0, 1), \tag{1.5}$$

we will show that the following result proved in [12] for $\delta_\varepsilon = \varepsilon$ still holds if $\delta_\varepsilon/\varepsilon^{\frac{3}{2}}$ tends to infinity

Theorem 1.1 *If u_ε is a sequence in $H^1(\Omega_\varepsilon)^3$ which satisfies the slip condition $u_\varepsilon\nu = 0$ on Γ_ε (ν denotes the unit outward normal to Ω_ε on Γ_ε) and it is such that $\|u_\varepsilon\|_{H^1(\Omega_\varepsilon)^3}$ is bounded,* *then the weak limit* $u = (u_1, u_2, u_3)$ *of* u_ε *in* $H^1(\Omega)^3$ *(which exists at least for a subsequence) satisfies*

$$u_3(x_1, x_2, 0) = 0, \quad \sum_{i=1}^{2} u_i(x_1, x_2, 0)\partial_i \Psi(y_1, y_2) = 0 \quad a.e. \ (x_1, x_2, y_1, y_2) \in \omega \times \mathbb{R}^2.$$

Applying this result to a viscous fluid in Ω_ε and assuming (1.2), we deduce that the slip and no-slip conditions on Γ_ε are asymptotically equivalent in this case. However, we prove that this result does not hold if the limit λ of $\delta_\varepsilon/\varepsilon^{\frac{3}{2}}$ belongs to $(0, +\infty)$. Indeed, if u_ε is a solution of the Navier-Stokes system, satisfying Navier's law on Γ_ε

$$\begin{cases} -\mu\Delta u_\varepsilon + \nabla p_\varepsilon + (u_\varepsilon \cdot \nabla) u_\varepsilon = f \text{ in } \Omega_\varepsilon \\ \text{div } u_\varepsilon = 0 \text{ in } \Omega_\varepsilon \\ u_\varepsilon \nu = 0 \text{ on } \Gamma_\varepsilon, \quad \dfrac{\partial T u_\varepsilon}{\partial \nu} + \gamma T u_\varepsilon = 0 \text{ on } \Gamma_\varepsilon \\ u_\varepsilon = 0 \text{ on } \partial\Omega_\varepsilon \setminus \Gamma_\varepsilon \end{cases} \tag{1.6}$$

where T is the orthogonal projection on the tangent space to Γ_ε, $\mu > 0$, $\gamma \geq 0$, and f belongs to $L^{\frac{6}{5}}(\omega \times (-1, 1))^3$, then the weak limits $u = (u_1, u_2, u_3)$ of u_ε in $H^1(\Omega)^3$ and p of p_ε in $L^2(\Omega)$ (which exist at least for a subsequence) are also a solution of the corresponding Navier-Stokes system in Ω satisfying Navier's law on $\{x_3 = 0\}$

$$u_3 = 0 \ \text{ on } \{x_3 = 0\}, \quad -\frac{\partial(u_1, u_2)}{\partial x_3} + \gamma(u_1, u_2) + \lambda^2 R(u_1, u_2) = 0 \text{ on } \{x_3 = 0\}, \tag{1.7}$$

where R is a nonnegative symmetric squared matrix of order 2 which does not depend on λ. In this case we do not have the adherence condition in the limit but the rugosity is large enough to enlarge the friction coefficient in the limit from γI to $\gamma I + \lambda^2 R$. If (1.2) is satisfied then R is positive, and then, taking λ converging to infinity in (1.7) we recuperate the adherence condition $u = 0$ on $\{x_3 = 0\}$.

When $\delta_\varepsilon/\varepsilon^{\frac{3}{2}}$ converges to zero we prove that the rugosity is so small that it has not any effect in the limit problem. In this case, the boundary condition on $\{x_3 = 0\}$ satisfied in the limit problem is just

$$u_3 = 0 \ \text{ on } \{x_3 = 0\}, \quad -\frac{\partial(u_1, u_2)}{\partial x_3} + \gamma(u_1, u_2) = 0 \text{ on } \{x_3 = 0\}.$$

Besides obtaining the limit problem of (1.6) we get a corrector (i.e. a strong approximation) of u_ε and p_ε in $H^1(\Omega_\varepsilon)^3$ and $L^2(\Omega_\varepsilon)$ respectively.

The homogenization of problem (1.6) has also been studied in [9] for a very general choice of Ω_ε, in particular it is not imposed a periodic structure for $\partial\Omega_\varepsilon$. These results of

[9] can be applied for Ω_ε given by (1.8), giving the existence of a family of vector spaces $\{V(x_1,x_2)\}_{(x_1,x_2)\in\omega} \subset \{x_3 = 0\}$, a finite positive Borel measure τ in $\{x_3 = 0\}$, absolutely continuous respect to the capacity, and a symmetric positively Borel function A in $\{x_3 = 0\}$ valuated in the symmetric nonnegatives matrices, such that, up to a subsequence, u_ε converges weakly in $H^1(\Omega)^3$ to a solution u of the variational problem

$$\begin{cases} u \in H^1(\Omega)^3, \quad \operatorname{div} u = 0 \text{ in } \Omega, \quad u(x_1,x_2,0) \in V(x_1,x_2) \text{ in } \omega, \quad \displaystyle\int_{\{x_3=0\}} |u|^2 d\tau < +\infty \\ \mu \displaystyle\int_\Omega Du : Dv\,dx + \int_\Omega [(u\cdot\nabla)u]v\,dx + \gamma \int_{\{x_3=0\}} uv\,dx + \int_{\{x_3=0\}} Auv\,d\tau = \int_\Omega fv\,dx, \\ \quad \forall v \in H^1(\Omega)^3, \quad \operatorname{div} v = 0 \text{ in } \Omega, \quad v(x_1,x_2,0) \in V(x_1,x_2) \text{ in } \omega, \quad \displaystyle\int_{\{x_3=0\}} |v|^2 d\tau < +\infty. \end{cases}$$

The results of the present paper give the value of the spaces $V(x_1,x_2)$, the measure τ and the matrix A according to the value of $\lambda = \lim_{\varepsilon\to 0} \delta_\varepsilon/\varepsilon^{\frac{3}{2}}$:

- If $\lambda = +\infty$, then $V(x_1,x_2) = Span(\{\nabla\Psi(y') : \; y' \in \mathbb{R}^2\})^\perp \times \{0\}$, $A \equiv 0$.
- If $\lambda \in (0,+\infty)$, then $V(x_1,x_2) = \mathbb{R}^2 \times \{0\}$, τ is the usual surface measure corresponding to the plane boundary $\{x_3 = 0\}$ and $A(x_1,x_2,0)\xi = \lambda^2 R(x_1,x_2,0)\xi$ for $\xi \in \mathbb{R}^2 \times \{0\}$.
- If $\lambda = 0$, then $V(x_1,x_2) = \mathbb{R}^2 \times \{0\}$, $A \equiv 0$.

So, our work provides in particular an example where the measure τ and the matrix A, whose existence is proved in [9], are different of zero. Another example giving A and τ different of zero has been obtained in [7] and [8], where analogously to (1.8), Ω_ε is defined by

$$\Omega_\varepsilon = \left\{ x = (x_1,x_2,x_3) \in \omega \times \mathbb{R} : \; -\varepsilon\Psi\left(\frac{x_1}{\varepsilon}\right) < x_3 < 1 \right\}, \tag{1.8}$$

with Ψ nonnegative, periodic of period 1, and not constant. In this case $V(x_1,x_2) = \{0\} \times \mathbb{R} \times \{0\}$, τ is the usual surface measure corresponding to $\{x_3 = 0\}$ and

$$A(x_1,x_2,0)\xi = \gamma \left(\int_0^1 \sqrt{1 + |\Psi'(y)|^2}dy - 1 \right) \xi, \quad \text{for } \xi \in \{0\} \times \mathbb{R} \times \{0\}.$$

We remark that this example has a different nature of the one given in the present paper. Indeed, in [7] and [8] the terms A and μ which appear in the limit problem of (1.6) are due to the fact that the surface element measure $\sqrt{1 + |\Psi'(\frac{x_1}{\varepsilon})|^2}dx_1dx_2$, corresponding to the oscillating boundary in Ω_ε, does not converge to the surface element dx_1dx_2 on $\{x_3 = 0\}$ but to $\int_0^1 \sqrt{1 + |\Psi'(y)|^2}dy\,dx_1dx_2$. However, in the case where Γ_ε is described by (1.4), the surface element $\sqrt{1 + \frac{\delta_\varepsilon^2}{\varepsilon^2}|\Psi'(\frac{x_1}{\varepsilon})|^2}dx_1dx_2$ converges to dx_1dx_2.

The proof of the results corresponding to the present paper is based on an original adaptation of the unfolding method ([5], [10], [14]) which is very related to the two-scale convergence method ([1], [17], [18]). The unfolding method is a very efficient tool to study periodic homogenization problems where the size of the periodic cell tends to zero. The idea is to introduce suitable changes of variables which transform every periodic cell into a simpler reference set by using a supplementary variable (microscopic variable).

Although our main interest in the present paper is to study the asymptotic behavior of (1.6), with Ω_ε described by (1.8) and $\delta_\varepsilon/\varepsilon$ converging to zero, in the last section of the paper we complete the work showing that if the limit of $\delta_\varepsilon/\varepsilon$ is strictly positive (and possibly $+\infty$), then Theorem 1.1 still holds. The proof of this result is also obtained by using the unfolding method.

The results of the present paper can be extended for dimension 2 using essentially the same proof. In fact, the results obtained for the Stokes problem hold in arbitrary dimension (taking the second members converging in the convenient spaces). It is important to mention that the exponent 3/2 which appears in the critical case does not depend on the dimension.

As we have mentioned above, the present paper is devoted to study the asymptotic behavior of a viscous fluid near a periodic oscillating boundary on which we consider slip conditions. A related problem was considered in [2] assuming that the solution is periodic with the same period of the boundary. We also refer to [3] for the case of Fourier's conditions. In both works [2] and [3] the boundary is supposed to be described by (1.1).

1.2 Notation

The elements $x \in \mathbb{R}^3$ will be decomposed as $x = (x', x_3)$ with $x' \in \mathbb{R}^2$, $x_3 \in \mathbb{R}$.

By Y', we denote the unitary cube of $\mathbb{R}^2$, $Y' = (-\frac{1}{2}, \frac{1}{2})^2$, and by $\widehat{Q}$ the set $\widehat{Q} = Y' \times (0, +\infty)$. For every $M > 0$ we write $\widehat{Q}_M = Y' \times (0, M)$.

We use the index $\sharp$ to mean periodicity with respect Y', for example $L^2_\sharp(Y')$ denotes the space of functions $u \in L^2_{loc}(\mathbb{R}^2)$ which are Y'-periodic, while $L^2_\sharp(\widehat{Q})$ denotes the space of functions $\widehat{u} \in L^2_{loc}(\mathbb{R}^2 \times (0, +\infty))$ such that

$$\int_{\widehat{Q}} |\widehat{u}|^2 \, dy < +\infty, \qquad \widehat{u}(y' + k', y_3) = \widehat{u}(y), \ \forall \, k' \in \mathbb{Z}^2, \text{ a.e. } y \in \mathbb{R}^2 \times (0, +\infty).$$

For a bounded measurable set $\Theta \subset \mathbb{R}^N$, we denote by $L^2_0(\Theta)$ the space of functions of $L^2(\Theta)$ with zero mean value in Θ.

We denote by ε and δ_ε two positive parameters devoted to tend to zero such that

$$\lim_{\varepsilon \to 0} \delta_\varepsilon = 0, \qquad \lim_{\varepsilon \to 0} \frac{\delta_\varepsilon}{\varepsilon} = 0.$$

For a fixed bounded Lipschitz open set $\omega \subset \mathbb{R}^2$ and a function $\Psi \in W^{2,\infty}_\sharp(Y')$, $\Psi \geq 0$ in Y', we denote

$$\Omega = \omega \times (0,1), \quad \Omega_\varepsilon = \Big\{ x \in \mathbb{R}^3 : \ x' \in \omega, \ -\delta_\varepsilon \Psi(\frac{x'}{\varepsilon}) < x_3 < 1 \Big\}$$

$$\Gamma = \omega \times \{0\}, \qquad \Gamma_\varepsilon = \Big\{ x \in \mathbb{R}^3 : \ x' \in \omega, \ x_3 = -\delta_\varepsilon \Psi(\frac{x'}{\varepsilon}) \Big\}.$$

We denote by ν the outside unitary orthogonal vector to Ω_ε on $\partial\Omega_\varepsilon$ and by T the orthogonal projection from $\mathbb{R}^3$ to $\{\nu\}^\perp$ (or equivalently, the tangent projection on Γ).

Our aim in the present paper is to study the asymptotic behavior of a viscous fluid in Ω_ε, which satisfies a slip boundary condition on Γ_ε. For this purpose we will use an adaptation of the method introduced in [5] to the study of periodic homogenization problems. It is now known as the unfolding method. We refer to [10], [13], [14], [17] for different applications of this method and its relation with the two-scale convergence method of G. Neguetseng and G. Allaire ([1], [18]).

In order to apply the unfolding method, we will need the following notation.

For $k' \in \mathbb{Z}^2$, we denote

$$C_\varepsilon^{k'} = \varepsilon k' + \varepsilon Y', \qquad \Omega_\varepsilon^{k'} = \Omega_\varepsilon \cap (C_\varepsilon^{k'} \times (-\infty, 1)).$$

We define $\kappa : \mathbb{R}^2 \to \mathbb{Z}^2$ by

$$\kappa(x') = k' \Leftrightarrow x' \in C_1^{k'}.$$

Remark that κ is well defined up to a set of zero measure in $\mathbb{R}^2$ (the set $\cup_{k' \in \mathbb{Z}^2} \partial C_1^{k'}$). Moreover, for every $\varepsilon > 0$, we have

$$\kappa(\frac{x'}{\varepsilon}) = k' \Leftrightarrow x' \in C_\varepsilon^{k'}.$$

For a.e. $x' \in \mathbb{R}^2$ we define $C_\varepsilon(x') = C_\varepsilon^{k'}$ such that $x' \in C_\varepsilon^{k'}$.

For every $\rho > 0$, we take

$$\omega_\rho = \{ x \in \omega : \mathrm{dist}(x, \partial\omega) > \rho \}$$

and

$$I_{\rho,\varepsilon} = \big\{ k' \in \mathbb{Z}^2 : C_\varepsilon^{k'} \cap \omega_\rho \neq \emptyset \big\}.$$

We define by $\mathcal{V}$ the space of functions $\widehat{v} : \mathbb{R}^2 \times (0, +\infty) \to \mathbb{R}$ such that

$$\widehat{v} \in H^1_\sharp(\widehat{Q}_M), \ \forall M > 0, \qquad \nabla \widehat{v} \in L^2_\sharp(\widehat{Q})^3.$$

It is a Hilbert space endowed with the norm $\|\cdot\|_{\mathcal{V}}$ defined by

$$\|\widehat{v}\|_{\mathcal{V}}^2 = \|\widehat{v}\|^2_{L^2(Y'\times\{0\})} + \|\nabla\widehat{v}\|^2_{L^2(\widehat{Q})^3}.$$

We denote by O_ε a generic real sequence which tends to zero with ε and can change from line to line.

We denote by C a generic positive constant which can change from line to line.

1.3 Main Results

In this section we describe the asymptotic behavior of a viscous fluid in the geometry Ω_ε described in Section 1.2 and satisfying slip conditions on Γ_ε. The proof of the corresponding results will be given in the next section.

Our first result is referred to the Stokes system. Namely, let us consider a sequence $(u_\varepsilon, p_\varepsilon) \in H^1(\Omega_\varepsilon)^3 \cap L^2(\Omega_\varepsilon)$, which satisfies

$$\begin{cases} -\Delta u_\varepsilon + \nabla p_\varepsilon = f_\varepsilon - div\, G_\varepsilon \text{ in } \Omega_\varepsilon \\ \text{div } u_\varepsilon = 0 \text{ in } \Omega_\varepsilon \\ u_\varepsilon \nu = 0 \text{ on } \Gamma_\varepsilon, \quad \dfrac{\partial T u_\varepsilon}{\partial \nu} - T G_\varepsilon \nu - T g_\varepsilon = 0 \text{ on } \Gamma_\varepsilon, \end{cases} \tag{1.9}$$

The second members $f_\varepsilon \in L^{\frac{6}{5}}(\Omega_\varepsilon)^3$, $G_\varepsilon \in L^2(\Omega_\varepsilon)^{3\times 3}$, $g_\varepsilon \in L^2(\Gamma_\varepsilon)^3$ are assumed to satisfy

$$\|f_\varepsilon\|_{L^{\frac{6}{5}}(\Omega_\varepsilon)^3} + \|G_\varepsilon\|_{L^2(\Omega_\varepsilon)^{3\times 3}} + \|g_\varepsilon\|_{L^2(\Gamma_\varepsilon)^3} \leq C, \quad \forall \varepsilon > 0, \tag{1.10}$$

and

$$\lim_{\varepsilon\to 0} \int_{\{x_3 < s\varepsilon\}} \left(|f_\varepsilon|^{\frac{6}{5}} + |G_\varepsilon|^2 \right) dx = 0, \quad \forall\, s > 0. \tag{1.11}$$

Remark 1.2 *Condition (1.10) implies that, up to a subsequence, there exist $f \in L^{\frac{6}{5}}(\Omega)^3$, $G \in L^2(\Omega)^{3\times 3}$, $g \in L^2(\Gamma)^3$ such that*

$$f_\varepsilon \rightharpoonup f \quad in\ L^{\frac{6}{5}}(\Omega)^3, \qquad G_\varepsilon \rightharpoonup G \quad in\ L^2(\Omega)^{3\times 3}, \quad g_\varepsilon\big(x', -\delta_\varepsilon \Psi(\frac{x'}{\varepsilon})\big) \rightharpoonup g \quad in\ L^2(\Gamma)^3. \tag{1.12}$$

Observe that we are not imposing any boundary conditions on $\partial\Omega_\varepsilon \setminus \Gamma_\varepsilon$ in (1.9). We are interested in the behavior of $(u_\varepsilon, p_\varepsilon)$ near Γ_ε and this does not depend on these boundary conditions. However these conditions are necessary to obtain an existence and uniqueness result of solution of problem (1.9). In this way, we have

Theorem 1.3 *We consider $f_\varepsilon \in L^{\frac{6}{5}}(\Omega_\varepsilon)^3$, $G_\varepsilon \in L^2(\Omega_\varepsilon)^{3\times 3}$ and $g_\varepsilon \in L^2(\Gamma_\varepsilon)^3$ which satisfy (1.10). Then, adding one of the following conditions*

i)

$$u_\varepsilon = 0 \text{ on } \partial\Omega_\varepsilon \setminus \Gamma_\varepsilon. \tag{1.13}$$

ii)

$$\begin{cases} \omega \text{ is a rectangle,} \\ u_\varepsilon,\ \dfrac{\partial u_\varepsilon}{\partial \nu} - p_\varepsilon \nu - G_\varepsilon \nu \text{ are periodic of period } \omega \text{ with respect to } x', \\ u_\varepsilon = 0 \text{ on } \{1\} \times \omega, \end{cases} \tag{1.14}$$

problem (1.9) has a unique solution $(u_\varepsilon, p_\varepsilon)$ in $H^1(\Omega_\varepsilon)^3 \times L^2_0(\Omega_\varepsilon)$. Moreover, there exists $C > 0$, which does not depend on ε, such that

$$\|u_\varepsilon\|_{H^1(\Omega_\varepsilon)^3} + \|p_\varepsilon\|_{L^2(\Omega_\varepsilon)} \le C. \tag{1.15}$$

Instead of supposing some boundary conditions on $\partial\Omega_\varepsilon \setminus \Gamma_\varepsilon$, Theorem 1.5 below describes the asymptotic behavior of a solution $(u_\varepsilon, p_\varepsilon)$ of (1.9) such that (1.15) holds. Indeed, next Proposition asserts that it is enough to assume the existence of $u_\varepsilon \in H^1(\Omega_\varepsilon)^3$, with $\|u_\varepsilon\|_{H^1(\Omega_\varepsilon)^3}$ bounded such that

$$\begin{cases} \displaystyle\int_{\Omega_\varepsilon} (Du_\varepsilon - G_\varepsilon) : Dv\, dx = \int_{\Omega_\varepsilon} f_\varepsilon v\, dx, \quad \forall\, v \in H^1_0(\Omega_\varepsilon)^3, \text{ with } \operatorname{div} v = 0 \text{ in } \Omega_\varepsilon \\ \operatorname{div}\ u_\varepsilon = 0 \text{ in } \Omega_\varepsilon \\ u_\varepsilon \nu = 0 \text{ on } \Gamma_\varepsilon, \quad \dfrac{\partial T u_\varepsilon}{\partial \nu} - T G_\varepsilon \nu - T g_\varepsilon = 0 \text{ on } \Gamma_\varepsilon. \end{cases}$$

Proposition 1.4 *We consider $f_\varepsilon \in L^{\frac{6}{5}}(\Omega_\varepsilon)^3$, $G_\varepsilon \in L^2(\Omega_\varepsilon)^{3\times 3}$ and $g_\varepsilon \in L^2(\Gamma_\varepsilon)^3$ which satisfy (1.10) and a sequence $u_\varepsilon \in H^1(\Omega_\varepsilon)^3$ such that $\|u_\varepsilon\|_{H^1(\Omega_\varepsilon)^3}$ is bounded and*

$$\int_{\Omega_\varepsilon} (Du_\varepsilon - G_\varepsilon) : Dv\, dx = \int_{\Omega_\varepsilon} f_\varepsilon v\, dx, \quad \forall\, v \in H^1_0(\Omega_\varepsilon)^3, \text{ with } div\, v = 0 \text{ in } \Omega_\varepsilon.$$

Then, there exists a unique $p_\varepsilon \in L^2_0(\Omega_\varepsilon)$, such that u_ε satisfies

$$-\Delta u_\varepsilon + \nabla p_\varepsilon = f_\varepsilon - div\, G_\varepsilon \text{ in } \Omega_\varepsilon.$$

Moreover $\|p_\varepsilon\|_{L^2_0(\Omega_\varepsilon)}$ is bounded.

Our main result referred to the asymptotic behavior of the solution of (1.9) is given by the following Theorem.

Theorem 1.5 *We consider $f_\varepsilon \in L^{\frac{6}{5}}(\Omega_\varepsilon)^3$, $G_\varepsilon \in L^2(\Omega_\varepsilon)^{3\times 3}$ and $g_\varepsilon \in L^2(\Gamma_\varepsilon)^3$ satisfying (1.10), (1.11) and such that there exist $f \in L^{\frac{6}{5}}(\Omega)^3$, $G \in L^2(\Omega)^{3\times 3}$, $g \in L^2(\Gamma)^3$, which satisfy (1.12). We also assume $(u_\varepsilon, p_\varepsilon)$ is a solution of (1.9) such that (1.15) holds. Then, there exists $(u,p) \in H^1(\Omega)^3 \times L^2_0(\Omega)$ such that, up to a subsequence,*

$$u_\varepsilon \rightharpoonup u \quad in\ H^1(\Omega)^3, \qquad p_\varepsilon \rightharpoonup p \quad in\ L^2(\Omega). \tag{1.16}$$

The pair (u,p) satisfies the Stokes system

$$\begin{cases} -\Delta u + \nabla p = f - div\, G \ \ in\ \Omega \\ div\, u = 0 \ \ in\ \Omega. \end{cases} \tag{1.17}$$

Moreover, denoting (this limit exists at least for a subsequence)

$$\lambda = \lim_{\varepsilon \to 0} \frac{\delta_\varepsilon}{\varepsilon^{\frac{3}{2}}} \in [0, +\infty], \tag{1.18}$$

it also satisfies the following boundary condition on Γ

i) If $\lambda = 0$, then

$$\begin{cases} u_3 = 0 \quad on\ \Gamma \\ -\partial_3 u' + (Ge_3)' - g' = 0 \quad on\ \Gamma. \end{cases} \tag{1.19}$$

ii) If $\lambda \in (0, +\infty)$, then defining $(\widehat{\phi}^i, \widehat{q}^i) \in \mathcal{V}^3 \times L^2_\sharp(\widehat{Q})$, $i = 1, 2$ as a solution of

$$\begin{cases} -\Delta \widehat{\phi}^i + \nabla \widehat{q}^i = 0 \quad in\ \mathbb{R}^2 \times \mathbb{R}^+ \\ div\, \widehat{\phi}^i = 0 \quad in\ \mathbb{R}^2 \times \mathbb{R}^+ \\ \widehat{\phi}^i_3 = \partial_i \Psi \quad on\ \mathbb{R}^2 \times \{0\} \\ -\partial_3 (\widehat{\phi}^i)' = 0 \quad on\ \mathbb{R}^2 \times \{0\}, \end{cases} \tag{1.20}$$

and $R \in \mathbb{R}^{2\times 2}$ by

$$R_{i,j} = \int_{\widehat{Q}} D\widehat{\phi}^i : D\widehat{\phi}^j \, dy, \quad \forall\, i,j \in \{1,2\} \tag{1.21}$$

we have

$$\begin{cases} u_3 = 0 \quad on\ \Gamma \\ -\partial_3 u' + \lambda^2 R u' + (Ge_3)' - g' = 0 \quad on\ \Gamma. \end{cases} \tag{1.22}$$

iii) If $\lambda = +\infty$, then defining

$$W = Span(\{(\nabla\Psi(y'), 0) : \; y' \in Y'\} \cup \{e_3\}), \tag{1.23}$$

and Q the orthogonal projection from $\mathbb{R}^3$ to $W^\perp$, we have

$$\begin{cases} u \in W^\perp \quad on\ \Gamma \\ -\partial_3 Qu + QGe_3 - Qg = 0 \quad on\ \Gamma. \end{cases} \tag{1.24}$$

The matrix R which appears in problem (1.22) is defined throughout a solution $(\widehat{\phi}^i, \widehat{q}^i)$ of (1.20). The following Proposition ensure the existence and uniqueness of solution of this problem. It also gives some smoothness properties for $(\widehat{\phi}^i, \widehat{q}^i)$. In particular it shows that $D\widehat{\phi}^i$ is uniquely defined and then that the matrix R is well defined.

Proposition 1.6 *There exists a unique solution $(\widehat{\phi}^i, \widehat{q}^i)$ of problem (1.22) in $\big((\mathcal{V}/\mathbb{R})^2 \times \mathcal{V}\big) \times L^2_\sharp(\widehat{Q})\big)$. Moreover, for every $r \in [2, +\infty)$, one has*

$$\|D\widehat{\phi}^i\|_{L^r(\widehat{Q})^{3\times 3}} + \|\widehat{q}^i\|_{L^r(\widehat{Q})} < +\infty. \tag{1.25}$$

Remark 1.7 *From Lemma 4 in [4] (see also [16] for related results) we can easily show that every solution $(\widehat{\phi}^i, \widehat{q}^i)$, $i = 1, 2$, of (1.20) is in $C^\infty_\sharp(\hat{Q})^3 \times C^\infty_\sharp(\hat{Q})$ and there exists a unique solution in $\big((\mathcal{V}/\mathbb{R})^2 \times \mathcal{V}\big) \times L^2_\sharp(\widehat{Q})\big)$. Moreover, for every $\alpha \in \mathbb{N}^N$ and every $z > 0$ there exist two positive constants $C_{z,\alpha}$, τ (the last one does not depend on α or z) such that*

$$|D^\alpha \widehat{\phi}^i|(y) + |D^\alpha \widehat{q}^i|(y) \le C_{z,\alpha} e^{-\tau y_3}, \quad \forall\, y \in \mathbb{R}^2 \times (z, +\infty).$$

Remark 1.8 *For $\lambda = 0$, the boundary conditions on Γ, (1.19), in the limit problem of (1.9) are the same we would find if Γ_ε does not vary with ε, i.e. in this case the rugosity of Γ_ε is very slight and the solution $(u_\varepsilon, p_\varepsilon)$ of (1.9) behaves as if Γ_ε coincides with the plane boundary Γ. For $0 < \lambda < +\infty$ (critical size), the boundary condition satisfied by the limit u of u_ε on the tangent space to Γ contains the new term $\lambda^2 Ru$. The effect of the rugosity of the wall Γ_ε is not worthless in this case. Finally, for $\lambda = +\infty$ the rugosity of Γ_ε is so strong that the limit u of u_ε does not only satisfies the condition $u_3 = 0$ on Γ, but it is also such that its tangent velocity on Γ, u', is orthogonal to the vectors $\nabla\Psi(y')$, for every $y' \in Y'$. In particular, if the linear space spanned by $\big\{\nabla\Psi(y') : \; y' \in Y'\big\}$ has dimension 2 (this holds if and only if Ψ is not constant in any straight line of $\mathbb{R}^2$, see [12]), we get that u satisfies the adherence condition $u = 0$ on Γ, i.e. although we have imposed a slip condition on Γ_ε, the rugosity forces u to satisfy a no-slip (adherence) condition on Γ. This result extends to the case where*

$$\lim_{\varepsilon \to 0} \frac{\delta_\varepsilon}{\varepsilon} = 0, \qquad \lim_{\varepsilon \to 0} \frac{\delta_\varepsilon}{\varepsilon^{\frac{3}{2}}} = +\infty,$$

the results obtained in [12] for $\delta_\varepsilon = \varepsilon$ (see also [6] for the nonperiodic case).

The limit equation (1.22) corresponding to the critical size $\lambda \in (0, +\infty)$ can be considered as the general one. In fact, if λ is tending to zero or $+\infty$ in (1.22) we get (1.19) and (1.24) respectively.

Remark 1.9 *If in Theorem 1.5, we also assume that one of the conditions (1.13) or (1.14) holds, then, assuming that there exists the limit λ given by (1.18), we deduce that (1.16) holds without extracting any subsequence. Moreover, if (1.13) is satisfied, then $u = 0$ on $\partial\Omega \setminus \Gamma$, while if (1.14) is satisfied, then u, $\frac{\partial u}{\partial \nu} - p\nu - G\nu$ are periodic with respect to x' and $u = 0$ on $\{1\} \times \omega$.*

Theorem 1.5 gives an approximation of $(u_\varepsilon, p_\varepsilon)$ in the weak topology of $H^1(\Omega)^3 \times L^2(\Omega)$. Assuming that the second members f_ε, G_ε and g_ε of (1.9) satisfy

$$|f_\varepsilon|^{\frac{6}{5}} \text{ is equiintegrable in } \Omega, \tag{1.26}$$

$$G_\varepsilon \text{ converges strongly to } G \text{ in } L^2(\Omega)^{3\times 3}, \tag{1.27}$$

let us now obtain an asymptotic expansion of $(Du_\varepsilon \chi_{\Omega_\varepsilon}, p_\varepsilon \chi_{\Omega_\varepsilon})$ which converges in the strong topology of $\mathbb{R}^2$ (corrector result).

Theorem 1.10 *We consider $f_\varepsilon \in L^{\frac{6}{5}}(\Omega_\varepsilon)^3$, $G_\varepsilon \in L^2(\Omega_\varepsilon)^{3\times 3}$ and $g_\varepsilon \in L^2(\Gamma_\varepsilon)^3$ which satisfy* (1.10), (1.11) *and (1.26) and are such that there exist $f \in L^{\frac{6}{5}}(\Omega)^3$, $G \in L^2(\Omega)^{3\times 3}$ and $g \in L^2(\Gamma)^3$ which satisfy* (1.12) *and* (1.27).

Let $(u_\varepsilon, p_\varepsilon) \in H^1(\Omega_\varepsilon)^3 \times L^2(\Omega_\varepsilon)$ be a solution of (1.9) *which satisfies* (1.15) *and it is such that there exists $(u, p) \in H^1(\Omega)^3 \times L^2(\Omega)$ which satisfies (1.16). We also assume that there exists the limit λ given by (1.18). Then, we have*

i)

$$\lim_{\varepsilon\to 0}\left(\int_{\Omega_\varepsilon\setminus\Omega} |u_\varepsilon|^2 dx + \int_\Omega |u_\varepsilon - u|^2 dx\right) = 0. \tag{1.28}$$

ii) If $\lambda = 0$ or $+\infty$, then, for every $\varphi \in C^1_c(\omega \times (-1, 1))$ we have

$$\lim_{\varepsilon\to 0}\left(\int_{\Omega_\varepsilon\setminus\Omega} |Du_\varepsilon|^2 \varphi\, dx + \int_\Omega |D(u_\varepsilon - u)|^2 \varphi\, dx\right) = 0, \tag{1.29}$$

$$\lim_{\varepsilon\to 0}\left(\int_{\Omega_\varepsilon\setminus\Omega} |p_\varepsilon|^2 \varphi\, dx + \int_\Omega |p_\varepsilon - p|^2 \varphi\, dx\right) = 0. \tag{1.30}$$

iii) If $\lambda \in (0,+\infty)$, then, taking $(\widehat{\phi}^i,\widehat{q}^i)$, $i=1,2$ as a solution of (1.20) *and defining $\widehat{u}:\omega\times(\mathbb{R}^2\times(0,+\infty))\to\mathbb{R}^3$ and $\widehat{p}:\omega\times(\mathbb{R}^2\times(0,+\infty))\to\mathbb{R}$ by*

$$\widehat{u}(x',y) = -\lambda u_1(x',0)\widehat{\phi}^1(y) - \lambda u_2(x',0)\widehat{\phi}^2(y), \tag{1.31}$$

$$\widehat{p}(x',y) = -\lambda u_1(x',0)\widehat{q}^1(y) - \lambda u_2(x',0)\widehat{q}^2(y), \tag{1.32}$$

for a.e. $(x',y)\in\omega\times(\mathbb{R}^2\times(0,+\infty))$, we have for every $\varphi\in C_c^1(\omega\times(-1,1))$

$$\lim_{\varepsilon\to 0}\left(\int_{\Omega_\varepsilon\setminus\Omega}|Du_\varepsilon|^2\varphi\,dx + \int_\Omega\left|Du_\varepsilon - Du - \frac{1}{\sqrt{\varepsilon}}D_y\widehat{u}(x',\frac{x}{\varepsilon})\right|^2\varphi\,dx\right) = 0, \tag{1.33}$$

$$\lim_{\varepsilon\to 0}\left(\int_{\Omega_\varepsilon\setminus\Omega}|p_\varepsilon|^2\varphi\,dx + \int_\Omega\left|p_\varepsilon - p - \frac{1}{\sqrt{\varepsilon}}\widehat{p}(x',\frac{x}{\varepsilon})\right|^2\varphi\,dx\right) = 0. \tag{1.34}$$

Remark 1.11 *If in Theorem 1.10, we also assume that $(u_\varepsilon,p_\varepsilon)$ satisfies one of the assumptions (1.13) or (1.14), then in (1.29), (1.30), (1.33) and (1.34), we can take $\varphi=1$.*

As a consequence of the results given above for the Stokes system, we can now describe the asymptotic behavior of the Navier-Stokes system posed in Ω_ε. To simplify the exposition we assume that the second member are fixed functions. The case of varying second members f_ε, G_ε and g_ε is analogous.

Theorem 1.12 *For $\mu>0$, $\gamma\geq 0$ and $f\in L^{\frac{6}{5}}(\omega\times(-1,1))^3$, we consider a solution $(u_\varepsilon,p_\varepsilon)\in H^1(\Omega_\varepsilon)^3\times L^2(\Omega_\varepsilon)$ of the Navier-Stokes system*

$$\begin{cases} -\mu\Delta u_\varepsilon + \nabla p_\varepsilon + (u_\varepsilon\cdot\nabla)u_\varepsilon = f \ \ in\ \Omega_\varepsilon \\ div\ u_\varepsilon = 0 \ \ in\ \Omega_\varepsilon \\ u_\varepsilon\nu = 0 \ \ on\ \Gamma_\varepsilon \\ \dfrac{\partial Tu_\varepsilon}{\partial\nu} + \gamma Tu_\varepsilon = 0 \ \ on\ \Gamma_\varepsilon, \end{cases} \tag{1.35}$$

which satisfies (1.15). Then, up to a subsequence, we have (1.16) with (u,p) solution of the Navier-Stokes system

$$\begin{cases} -\mu\Delta u + \nabla p + (u\cdot\nabla)u = f \ \ in\ \Omega \\ div\ u = 0 \ \ in\ \Omega. \end{cases} \tag{1.36}$$

Defining λ by (1.18) (this limit exists up to a subsequence), we also have *i) If* $\lambda = 0$*, then*

$$\begin{cases} u_3 = 0 \quad on\ \Gamma \\ -\partial_3 u' + \gamma u' = 0 \quad on\ \Gamma. \end{cases} \tag{1.37}$$

Moreover (1.29), (1.30) hold for every $\varphi \in C_c^1(\omega \times (-1,1))$*.*

ii) If $\lambda \in (0, +\infty)$*, then*

$$\begin{cases} u_3 = 0 \quad on\ \Gamma \\ -\partial_3 u' + \gamma u' + \lambda^2 R u' = 0 \quad on\ \Gamma, \end{cases} \tag{1.38}$$

with R *defined by (1.21). Moreover, defining* $\widehat{u}$*,* $\widehat{p}$ *by (1.31), (1.32) we have (1.33), (1.34) for every* $\varphi \in C_c^1(\omega \times (-1,1))$*.*

iii) If $\lambda = +\infty$*, then defining* W *by (1.23) and* Q *as the ortogonal projection from* $\mathbb{R}^3$ *onto* $W^\perp$ *we have*

$$\begin{cases} u \in W^\perp \quad on\ \Gamma \\ -\partial_3 Qu + \gamma Qu = 0 \quad on\ \Gamma. \end{cases} \tag{1.39}$$

Moreover, (1.29), (1.30) hold for every $\varphi \in C_c^1(\omega \times (-1,1))$*.*

Remark 1.13 *Assuming also that* $(u_\varepsilon, p_\varepsilon)$ *satisfies one of the boundary conditions given by (1.13) or (1.14) there exists at least a solution of the Navier-Stokes system (1.35). Moreover, this solution satisfies (1.15) and thus, Theorem 1.12 can be applied. In this case, we can take* $\varphi = 1$ *in (1.29), (1.30), (1.33), (1.34).*

1.4 Proof of the results of Section 1.3.

Proof of Theorem 1.3. It is a simple consequence of Proposition 1.4 and Lax-Milgram theorem. □

Proof of Proposition 1.4. It follows from Statement ii) in Proposition 1.14 below. □

Proposition 1.14 *There exists a constant* $c > 0$ *such that for every* $\varepsilon > 0$ *(small enough), we have*

i) There exists a linear continuous operator $L_\varepsilon : L_0^2(\Omega_\varepsilon) \to H_0^1(\Omega_\varepsilon)^3$ *with* $\|L_\varepsilon\| \leq c$*, such that*

$$div\, L_\varepsilon(h_\varepsilon) = h_\varepsilon \quad in\ \Omega_\varepsilon, \quad \forall\, h_\varepsilon \in L_0^2(\Omega_\varepsilon).$$

ii) If $\zeta_\varepsilon \in H^{-1}(\Omega_\varepsilon)^3$ *satisfies*

$$\langle \zeta_\varepsilon, v\rangle = 0, \quad \forall\, v \in H_0^1(\Omega_\varepsilon)^3, \text{ with } div\, v = 0 \ \text{ in } \Omega_\varepsilon, \tag{1.40}$$

then, there exists a unique $p_\varepsilon \in L_0^2(\Omega_\varepsilon)$ *with* $\zeta_\varepsilon = \nabla p_\varepsilon$ *in* Ω_ε*. Moreover*

$$\|p_\varepsilon\|_{L_0^2(\Omega_\varepsilon)} \le c\|\zeta_\varepsilon\|_{H^{-1}(\Omega_\varepsilon)^3}. \tag{1.41}$$

iii) $H^1(\Omega_\varepsilon)$ *is continuously injected in* $L^6(\Omega_\varepsilon)$ *with*

$$\|u_\varepsilon\|_{L^6(\Omega_\varepsilon)} \le c\|u_\varepsilon\|_{H^1(\Omega_\varepsilon)}, \quad \forall\, u_\varepsilon \in H^1(\Omega_\varepsilon). \tag{1.42}$$

Proof. Along the proof, we use the application $\eta_\varepsilon : \Omega_\varepsilon \to \Omega$ defined by

$$\eta_\varepsilon(x) = \left(x', 1 + \frac{x_3 - 1}{1 + \delta_\varepsilon \Psi(\frac{x'}{\varepsilon})}\right).$$

In order to prove i) we remark that using the change of variables $z = \eta_\varepsilon(x)$, the equation

$$\operatorname{div} u_\varepsilon = h_\varepsilon \ \text{ in } \Omega_\varepsilon,$$

is equivalent to

$$\operatorname{div}_z \tilde u_\varepsilon = \tilde h_\varepsilon(z) + \frac{\delta_\varepsilon}{\varepsilon}\frac{z_3 - 1}{1 + \delta_\varepsilon\Psi(\frac{z'}{\varepsilon})}\partial_{z_3}\tilde u'_\varepsilon(z)\nabla\Psi(\frac{z'}{\varepsilon}) + \delta_\varepsilon\frac{\Psi(\frac{z'}{\varepsilon})}{1 + \delta_\varepsilon\Psi(\frac{z'}{\varepsilon})}\partial_{z_3}\tilde u_{\varepsilon,3}(z) \ \text{ in } \Omega, \tag{1.43}$$

where we have denoted $\tilde u_\varepsilon(z) = u_\varepsilon \circ \eta_\varepsilon^{-1}(z)$, $\tilde h_\varepsilon(z) = h_\varepsilon \circ \eta_\varepsilon^{-1}(z)$.

Now, since Ω is Lipschitz, it is well known that there exists a linear continuous operator $L : L_0^2(\Omega) \to H_0^1(\Omega)^3$ such that

$$\operatorname{div} L(h) = h \ \text{ in } \Omega, \quad \forall\, h \in L_0^2(\Omega).$$

The Banach fixed point theorem implies that for ε small enough and every $h \in L_0^2(\Omega)$ the problem

$$u = L\left(h + \frac{\delta_\varepsilon}{\varepsilon}\frac{z_3 - 1}{1 + \delta_\varepsilon\Psi(\frac{z'}{\varepsilon})}\partial_3 u'\nabla\Psi(\frac{z'}{\varepsilon}) + \delta_\varepsilon\frac{\Psi(\frac{z'}{\varepsilon})}{1 + \delta_\varepsilon\Psi(\frac{z'}{\varepsilon})}\partial_3\tilde u_3\right), \quad u \in H_0^1(\Omega)^3,$$

has a unique solution which we denote by $R_\varepsilon h$. The operator R_ε defined in this way is linear and satisfies

$$\|R_\varepsilon h\|_{H_0^1(\Omega)^3} \le \|L\|\left(\|h\|_{L_0^2(\Omega)} + C\frac{\delta_\varepsilon}{\varepsilon}\|R_\varepsilon h\|_{H_0^1(\Omega)^3}\right),$$

i.e.

$$\|R_\varepsilon h\|_{H_0^1(\Omega)^3} \leq \frac{\|L\|}{1 - C\frac{\delta_\varepsilon}{\varepsilon}\|L\|}\|h\|_{L_0^2(\Omega)}, \quad \forall\, h \in L_0^2(\Omega),$$

where C only depends on Ψ.

This proves that R_ε is continuous with norm uniformly bounded for ε small enough. Taking into account that $\tilde{u}_\varepsilon = R_\varepsilon \tilde{h}_\varepsilon$ solves (1.43), we get the proof of i) just taking

$$L_\varepsilon h_\varepsilon = \big(R_\varepsilon(h_\varepsilon \circ \eta_\varepsilon^{-1})\big) \circ \eta_\varepsilon, \quad \forall\, h_\varepsilon \in L_0^2(\Omega_\varepsilon),$$

and using that η_ε, η_ε^{-1} are respectively bounded in $W^{1,\infty}(\Omega_\varepsilon)^3$ and $W^{1,\infty}(\Omega)^3$.

Let us now use statement i) to prove ii). For this purpose, given $\zeta_\varepsilon \in H^{-1}(\Omega_\varepsilon)^3$ which satisfies (1.40), we have that $\zeta_\varepsilon \circ L_\varepsilon \in L_0^2(\Omega_\varepsilon)'$ and thus, there exists $p_\varepsilon \in L_0^2(\Omega_\varepsilon)$ which satisfies

$$\|p_\varepsilon\|_{L_0^2(\Omega_\varepsilon)} = \|\zeta_\varepsilon \circ L_\varepsilon\|_{L_0^2(\Omega_\varepsilon)'} \leq c\|\zeta_\varepsilon\|_{H^{-1}(\Omega_\varepsilon)^3}$$

and

$$\langle \zeta_\varepsilon, L_\varepsilon h_\varepsilon \rangle = -\int_{\Omega_\varepsilon} p_\varepsilon h_\varepsilon \, dx, \quad \forall\, h_\varepsilon \in L_0^2(\Omega_\varepsilon).$$

Since ζ_ε vanishes on the functions with zero divergence, we have

$$\langle \zeta_\varepsilon, u_\varepsilon \rangle = \langle \zeta_\varepsilon, L_\varepsilon(\operatorname{div} u_\varepsilon) \rangle = -\int_{\Omega_\varepsilon} p_\varepsilon \operatorname{div} u_\varepsilon \, dx, \quad \forall\, u_\varepsilon \in H_0^1(\Omega_\varepsilon)^3.$$

This proves that $\nabla p_\varepsilon = \zeta_\varepsilon$ in Ω_ε. The uniqueness of p_ε is immediate from the fact that every distribution with zero gradient in Ω_ε is constant. This finishes the proof of ii).

The proof of iii) is immediate using that $H^1(\Omega)$ is continuously imbedded in $L^6(\Omega)$ and the change of variable $z = \eta_\varepsilon(x)$ which transforms Ω_ε in Ω. □

In order to prove Theorem 1.5 let us introduce an adaptation of the unfolding method (see e.g. [5], [10], [13], [14], [17]), which is strongly related to the two-scale convergence method ([1], [18]). For this purpose, given $(u_\varepsilon, p_\varepsilon) \in H^1(\Omega_\varepsilon)^3 \times L_0^2(\Omega_\varepsilon)$ and $\rho > 0$, we define $(\widehat{u}_\varepsilon, \widehat{p}_\varepsilon)$ by

$$\widehat{u}_\varepsilon(x', y) = u_\varepsilon(\varepsilon\kappa(\frac{x'}{\varepsilon}) + \varepsilon y', \varepsilon y_3) \tag{1.44}$$

$$\widehat{p}_\varepsilon(x', y) = p_\varepsilon(\varepsilon\kappa(\frac{x'}{\varepsilon}) + \varepsilon y', \varepsilon y_3), \tag{1.45}$$

for a.e. $(x', y') \in \omega_\rho \times \widehat{Y}_\varepsilon$, with

$$\widehat{Y}_\varepsilon = \{y \in Y' \times \mathbb{R} : -\frac{\delta_\varepsilon}{\varepsilon}\Psi(y') < y_3 < 1/\varepsilon\}.$$

Remark 1.15 *For $k' \in \mathbb{Z}^2$ the restriction of $(\widehat{u}_\varepsilon, \widehat{p}_\varepsilon)$ to $C_\varepsilon^{k'} \times \widehat{Y}_\varepsilon$ does not depend on x', while as function of y it is obtained from $(u_\varepsilon, p_\varepsilon)$ by using the change of variables*

$$y' = \frac{x' - \varepsilon k'}{\varepsilon}, \qquad y_3 = \frac{x_3}{\varepsilon}, \tag{1.46}$$

which transforms $\Omega_\varepsilon^{k'}$ into $\widehat{Y}_\varepsilon$.

In order to study the asymptotic behavior of $(u_\varepsilon, p_\varepsilon)$ near Γ_ε, let us study the asymptotic behavior of $(\widehat{u}_\varepsilon, \widehat{p}_\varepsilon)$ in $\omega_\rho \times \widehat{Q}_M$, for every $M > 0$. We will need the following previous Lemma.

Lemma 1.16 *Let $v_\varepsilon \in L^2(\omega)$ be a sequence which converges weakly to a function v in $L^2(\omega)$. For ρ, we define $\bar{v}_\varepsilon \in L^2(\omega_\rho)$ by*

$$\bar{v}_\varepsilon(x') = \frac{1}{\varepsilon^2} \int_{C_\varepsilon(x')} v_\varepsilon(z')\, d\, z', \quad a.e.\ x' \in \omega_\rho.$$

Then we have:

i) For every $\tau' \in \mathbb{R}^2$, the sequence w_ε defined by

$$w_\varepsilon(x') = \frac{\bar{v}_\varepsilon(x' + \varepsilon\tau') - \bar{v}_\varepsilon(x')}{\sqrt{\varepsilon}}$$

converges to zero in the sense of distributions in ω_ρ.

ii) If the convergence of v_ε is strong, then $\bar{v}_\varepsilon$ converges strongly to v in $L^2(\omega_\rho)$.

Proof. In order to prove i), we use that for every $\varphi \in C_c^\infty(\omega_\rho)$ and $\varepsilon > 0$, small enough, we have

$$\int_{\omega_\rho} \frac{\bar{v}_\varepsilon(x' + \varepsilon\tau') - \bar{v}_\varepsilon(x')}{\sqrt{\varepsilon}} \varphi(x') dx' = \int_{\omega_\rho} \frac{v_\varepsilon(z')}{\sqrt{\varepsilon}} \left(\frac{1}{\varepsilon^2} \int_{C_\varepsilon(z')} [\varphi(x' - \varepsilon\tau') - \varphi(x')] dx' \right) dz',$$

where, since φ is Lipschitz, the right-hand side tends to zero.

Statement ii) easily follows using that the sequence $v_\varepsilon^m \in L^2(\omega_\rho)$ defined by

$$v_\varepsilon^m(x') = \frac{1}{\varepsilon^2} \int_{C_\varepsilon(x')} v(z')\, d\, z', \quad \text{a.e. } x' \in \omega_\rho$$

converges strongly to v in $L^2(\omega_\rho)$ and the inequality

$$\|\bar{v}_\varepsilon - v_\varepsilon^m\|_{L^2(\omega_\rho)} \leq \|v_\varepsilon - v\|_{L^2(\Omega)}.$$

□

The following Lemmas describe the asymptotic behavior of $(\widehat{u}_\varepsilon, \widehat{p}_\varepsilon)$ given by (1.44), (1.45), when $(u_\varepsilon, p_\varepsilon) \in H^1(\Omega_\varepsilon)^3 \times L_0^2(\Omega_\varepsilon)$ satisfies (1.15).

Lemma 1.17 *Let $p_\varepsilon \in L^2_0(\Omega_{\varepsilon,\rho})$ be with bounded norm. Then, up to a subsequence, there exists $\widehat{p} \in L^2(\omega \times \widehat{Q})$ such that the sequence $\widehat{p}_\varepsilon$ defined by (1.45) satisfies*

$$\sqrt{\varepsilon}\widehat{p}_\varepsilon \rightharpoonup \widehat{p} \text{ in } L^2(\omega_\rho \times \widehat{Q}_M),\ \forall\, M, \rho > 0. \tag{1.47}$$

Proof. For every $\rho, M > 0$, the definition of $\widehat{p}_\varepsilon$ proves

$$\left\{\begin{aligned} &\int_{\omega_\rho \times \widehat{Q}_M} |\sqrt{\varepsilon}\widehat{p}_\varepsilon|^2\,dx'\,dy \le \sum_{k' \in I_{\rho,\varepsilon}} \varepsilon^3 \int_{\widehat{Q}_M} |p_\varepsilon(\varepsilon(k'+y'), \varepsilon y_3)|^2\,dy \\ &\le \sum_{k' \in I_{\rho,\varepsilon}} \int_{\Omega_\varepsilon^{k'}} |p_\varepsilon(x)|^2\,dx \le \int_{\Omega_\varepsilon} |p_\varepsilon|^2\,dx. \end{aligned}\right. \tag{1.48}$$

Thus, $\sqrt{\varepsilon}\widehat{p}_\varepsilon$ is bounded in $L^2(\omega_\rho \times \widehat{Q}_M)$ for every $\rho, M > 0$, and so, using a diagonal procedure, there exists $\widehat{p}$ such that (1.47) holds. Taking into account in (1.48) the semicontinuity of the norm for the weak convergence, we deduce

$$\int_{\omega_\rho \times \widehat{Q}_M} |\widehat{p}|^2 dx'\,dy \le \liminf_{\varepsilon \to 0} \int_{\Omega_\varepsilon} |p_\varepsilon|^2\,dx,$$

for every $\rho, M > 0$ and thus, the monotone convergence theorem shows that $\widehat{p}$ belongs to $L^2(\omega \times \widehat{Q})$. □

Lemma 1.18 *We consider a sequence $u_\varepsilon \in H^1(\Omega_\varepsilon)^3$ with bounded norm, such that $u_\varepsilon \nu = 0$ on Γ_ε and such that (it always holds for a subsequence) there exists $u \in H^1(\Omega)^3$ with u_ε converging weakly to u in $H^1(\Omega)^3$. Then, the third component u_3 of u vanishes on Γ.*

Moreover, if we also assume that there exists the limit λ given by (1.18) and that λ belongs to $(0, +\infty]$, we have

i) If $\lambda = +\infty$, then

$$u'(x', 0)\nabla\Psi(y') = 0 \ \text{ a.e. } (x', y') \in \omega \times Y'. \tag{1.49}$$

ii) If $\lambda \in (0, +\infty)$, then there exists $\widehat{u} \in L^2(\Omega; \mathcal{V}^3)$ with

$$\widehat{u}_3(x', y', 0) = -\lambda\nabla\Psi(y')u'(x', 0), \quad \text{a.e. } (x', y') \in \omega \times Y', \tag{1.50}$$

such that for every $\rho, M > 0$, the sequence $\widehat{u}_\varepsilon$ defined by (1.44) satisfies

$$\frac{1}{\sqrt{\varepsilon}}\widehat{u}_\varepsilon \rightharpoonup \widehat{u} \text{ in } L^2(\omega_\rho; H^1(\widehat{Q}_M)^3). \tag{1.51}$$

Besides, if $\operatorname{div} u_\varepsilon = 0$ in Ω, then

$$\operatorname{div}_y \widehat{u} = 0 \ \text{ in } \omega \times \widehat{Q}. \tag{1.52}$$

Proof.
Step 1. Let us first prove that u_3 vanishes on Γ.

Since $u_\varepsilon \nu = 0$ on Γ_ε, for every $\varphi \in C^1_c(\omega \times (-1,1))$ we have

$$\int_{\Omega_\varepsilon} u_\varepsilon \nabla\varphi \, dx = -\int_{\Omega_\varepsilon} \operatorname{div} u_\varepsilon \, \varphi \, dx. \tag{1.53}$$

Using

$$\left|\int_{\Omega_\varepsilon\setminus\Omega} u_\varepsilon \nabla\varphi \, dx\right| \le \left(\int_{\Omega_\varepsilon} |u_\varepsilon|^2 \, dx\right)^{\frac{1}{2}} \left(\int_{\Omega_\varepsilon\setminus\Omega} |\nabla\varphi|^2 \, dx\right)^{\frac{1}{2}} \to 0,$$

$$\left|\int_{\Omega_\varepsilon\setminus\Omega} \operatorname{div} u_\varepsilon \varphi \, dx\right| \le \left(\int_{\Omega_\varepsilon} |\operatorname{div} u_\varepsilon|^2 \, dx\right)^{\frac{1}{2}} \left(\int_{\Omega_\varepsilon\setminus\Omega} |\varphi|^2 \, dx\right)^{\frac{1}{2}} \to 0,$$

and the weak convergence of u_ε to u in $H^1(\Omega)^3$ we can pass to the limit in (1.53) to deduce

$$\int_{\Omega} u \nabla\varphi \, dx = -\int_{\Omega} \operatorname{div} u \, \varphi \, dx,$$

and then

$$\int_{\Gamma} u_3 \varphi \, dx' = 0, \quad \forall\, \varphi \in C^1_c(\omega \times (-1,1)),$$

which proves $u_3 = 0$ on Γ.

Step 2. Let us obtain some estimates for the sequence $\widehat{u}_\varepsilon$ given by (1.44).

For $\rho, M > 0$, the definition (1.44) of $\widehat{u}_\varepsilon$ proves for every $\varepsilon > 0$ small enough

$$\left\{\begin{array}{l} \displaystyle\int_{\omega_\rho \times \widehat{Q}_M} |D_y \widehat{u}_\varepsilon(x',y)|^2 dx' dy \le \varepsilon^4 \sum_{k' \in I_{\rho,\varepsilon}} \int_{Y' \times (0,M)} |Du_\varepsilon(\varepsilon(k'+y'), \varepsilon y_3)|^2 dy \\ \displaystyle\le \sum_{k' \in I_{\rho,\varepsilon}} \varepsilon \int_{\Omega_\varepsilon^{k'}} |Du_\varepsilon|^2 dx \le \varepsilon \int_{\Omega_\varepsilon} |Du_\varepsilon|^2 dx \le C\varepsilon. \end{array}\right. \tag{1.54}$$

On the other hand, defining

$$\bar{u}_\varepsilon(x') = \frac{1}{\varepsilon^2} \int_{C_\varepsilon(x')} u_\varepsilon(\tau', 0) \, d\tau = \int_{Y'} \widehat{u}_\varepsilon(x', y', 0) \, dy', \tag{1.55}$$

and using the inequality

$$\int_{\widehat{Q}_M} |\widehat{u}_\varepsilon(x',y) - \bar{u}_\varepsilon(x')|^2 \, dy \le C_M \int_{\widehat{Q}_M} |\nabla_y \widehat{u}_\varepsilon|^2 \, dy, \quad \text{a.e. } x' \in \omega_\rho, \tag{1.56}$$

where C_M does not depend on ε and taking into account(1.54), we deduce that

$$\widehat{U}_\varepsilon = \frac{\widehat{u}_\varepsilon(x', y) - \bar{u}_\varepsilon}{\sqrt{\varepsilon}} \text{ is bounded in } L^2(\omega_\rho; H^1(\widehat{Q}_M)^3), \quad \forall \rho, M > 0. \tag{1.57}$$

Thus, there exists $\widehat{u} : \omega \times \widehat{Q} \to \mathbb{R}^3$, such that, up to a subsequence,

$$\widehat{U}_\varepsilon \rightharpoonup \widehat{u} \text{ in } L^2(\omega_\rho; H^1(\widehat{Q}_M)^3), \quad \forall\, \rho, M > 0, \tag{1.58}$$

and then

$$\frac{1}{\sqrt{\varepsilon}} D_y \widehat{u}_\varepsilon \rightharpoonup D_y \widehat{u} \text{ in } L^2(\omega_\rho \times Q_M)^{3\times 3}, \quad \forall\, \rho, M > 0. \tag{1.59}$$

Passing to the limit by semicontinuity in inequalities (1.54) and (1.56) (this latest one after integration in ω_ρ), we get

$$\int_{\omega_\rho \times \widehat{Q}_M} |D_y \widehat{u}|^2 dx' dy \leq C, \qquad \int_{\omega_\rho \times \widehat{Q}_M} |\widehat{u}|^2 dx' dy \leq C_M,$$

and then, by the arbitrariness of ρ and M

$$\widehat{u} \in L^2(\omega; \mathcal{V}^3). \tag{1.60}$$

Moreover, if we also assume that $\operatorname{div} u_\varepsilon = 0$ in Ω_ε, then by definition (1.44) of $\widehat{u}_\varepsilon$, we have $\operatorname{div}_y \widehat{u}_\varepsilon = 0$ in $\omega_\rho \times \widehat{Q}_M$, which together to (1.59) proves

$$\operatorname{div}_y \widehat{u} = 0 \text{ in } \omega \times \widehat{Q}. \tag{1.61}$$

Step 3. Let us prove that $\widehat{u}$ is Y'-periodic in y'.

We observe that by definition (1.44) of $\widehat{u}_\varepsilon$, for every $\rho, M > 0$, we have

$$\widehat{u}_\varepsilon(x_1 + \varepsilon, x_2, -\frac{1}{2}, y_2, y_3) = \widehat{u}_\varepsilon(x', \frac{1}{2}, y_2, y_3), \text{ a.e. } (x', y_2, y_3) \in \omega_\rho \times (-\frac{1}{2}, \frac{1}{2}) \times (0, M).$$

Therefore the sequence $\widehat{U}_\varepsilon$ satisfies

$$\widehat{U}_\varepsilon(x_1 + \varepsilon, x_2, -\frac{1}{2}, y_2, y_3) - \widehat{U}_\varepsilon(x', \frac{1}{2}, y_2, y_3) = \frac{-\bar{u}_\varepsilon(x_1 + \varepsilon, x_2) + \bar{u}_\varepsilon(x')}{\sqrt{\varepsilon}} \tag{1.62}$$

By (1.55) and u_ε bounded in $L^2(\Gamma)^3$ we can apply Lemma 1.16 i) to deduce that the right-hand side of this equality tends to zero in the sense of distributions in ω_ρ. Therefore, passing to the limit in (1.62) by (1.58), and taking into account the arbitrariness of ρ and M we get

$$\widehat{u}(x', -\frac{1}{2}, y_2, y_3) - \widehat{u}(x', \frac{1}{2}, y_2, y_3) = 0 \text{ a.e. } (x', y_2, y_3) \in \omega_\rho \times (-\frac{1}{2}, \frac{1}{2}) \times \mathbb{R}.$$

Analogously, we can prove

$$\widehat{u}(x', y_1, -\frac{1}{2}, y_3) - \widehat{u}(x', y_1, \frac{1}{2}, y_3) = 0 \text{ a.e. } (x', y_1, y_3) \in \omega_\rho \times (-\frac{1}{2}, \frac{1}{2}) \times \mathbb{R}.$$

These equalities prove that $\widehat{u}$ is periodic with respect to Y'.

Step 4. Using the compact embedding of $H^1(\Omega)$ into $L^2(\Gamma)$ and Lemma 1.16 ii), we have that $\bar{u}_\varepsilon$ converges strongly to $u(x', 0)$ in $L^2(\omega_\rho)^3$, for every $\rho > 0$. Thus, by (1.57), we deduce

$$\widehat{u}_\varepsilon(x', y) \to u(x', 0) \text{ in } L^2(\Omega_\rho; H^1(\widehat{Q}_M)^3), \quad \forall M, \rho > 0. \tag{1.63}$$

Step 5. For $\rho > 0$, using the change of variables (1.46), which defines $\widehat{u}_\varepsilon$, in the equality $u_\varepsilon \nu = 0$ on Γ_ε, we get

$$-\frac{\delta_\varepsilon}{\varepsilon}\nabla\Psi(y')\widehat{u}'_\varepsilon(x', y', -\frac{\delta_\varepsilon}{\varepsilon}\Psi(y')) - \widehat{u}_{\varepsilon,3}(x', y', -\frac{\delta_\varepsilon}{\varepsilon}\Psi(y')) = 0 \text{ a.e. in } \omega_\rho \times Y'. \tag{1.64}$$

Thanks to (1.64) and (1.54), we have then

$$\left|\frac{\delta_\varepsilon}{\varepsilon}\nabla\Psi(y')\widehat{u}'_\varepsilon(x', y', 0) + \widehat{u}_{\varepsilon,3}(x', y', 0)\right| =$$
$$\int_{-\frac{\delta_\varepsilon}{\varepsilon}\Psi(y')}^0 \left|\frac{\delta_\varepsilon}{\varepsilon}\nabla\Psi(y')\partial_3\widehat{u}'_\varepsilon(x', y', t) + \partial_3\widehat{u}_{\varepsilon,3}(x', y', t)\right| dt$$
$$\leq C\left(\frac{\delta_\varepsilon}{\varepsilon}\right)^{\frac{1}{2}}\left(\int_{-\frac{\delta_\varepsilon}{\varepsilon}\Psi(y')}^0 |\partial_3\widehat{u}_\varepsilon(x', y', t)|^2\, dt\right)^{\frac{1}{2}} \quad \text{a.e. } (x', y') \in \omega_\rho \times Y'.$$

Taking the power two, integrating in $\omega_\rho \times Y'$ and using (1.54) we then deduce

$$\int_{\omega_\rho \times Y'} \left|\frac{\delta_\varepsilon}{\varepsilon}\nabla\Psi(y')\widehat{u}'_\varepsilon(x', y', 0) + \widehat{u}_{\varepsilon,3}(x', y', 0)\right|^2 dx'dy' \leq C\delta_\varepsilon,$$

which implies

$$\int_{\omega_\rho \times Y'} \left|\frac{\delta_\varepsilon}{\varepsilon}\nabla\Psi(y')\widehat{u}'_\varepsilon(x', y', 0) + \widehat{u}_{\varepsilon,3}(x', y', 0) \right.$$
$$\left. - \int_{Y'}\left(\frac{\delta_\varepsilon}{\varepsilon}\nabla\Psi(z')\widehat{u}'_\varepsilon(x', z', 0) + \widehat{u}_{\varepsilon,3}(x', z', 0)\right) dz'\right|^2 dx'dy' \leq C\delta_\varepsilon.$$

Dividing by ε, and taking into account that $\nabla\Psi$ has mean value zero in Y', we get

$$\left\{\begin{array}{l}\displaystyle\int_{\omega_\rho\times Y'}\left|\frac{\delta_\varepsilon}{\varepsilon^{3/2}}\nabla\Psi(y')\widehat{u}'_\varepsilon(x',y',0)-\frac{\delta_\varepsilon}{\varepsilon}\int_{Y'}\nabla\Psi(z')\left(\frac{\widehat{u}'_\varepsilon(x',z',0)-\bar{u}'_\varepsilon(x')}{\sqrt{\varepsilon}}\right)dz'\right.\\ \displaystyle\qquad\qquad\qquad\left.+\frac{\widehat{u}_{\varepsilon,3}(x',y',0)-\bar{u}_{\varepsilon,3}(x')}{\sqrt{\varepsilon}}\right|^2 dx'dy'\leq C\frac{\delta_\varepsilon}{\varepsilon}\to 0,\quad \forall\,\rho>0.\end{array}\right. \tag{1.65}$$

Depending on the values of λ, we deduce:

If $\lambda=+\infty$, statement (1.65) shows that $\frac{\delta_\varepsilon}{\varepsilon^{3/2}}\nabla\Psi(y')\widehat{u}'_\varepsilon(x',y',0)$ is bounded in $L^2(\omega_\rho\times Y')$, for every $\rho>0$ and then that $\nabla\Psi(y')\widehat{u}'_\varepsilon(x',y',0)$ tends to zero in $L^2(\omega_\rho\times Y')$, for every $\rho>0$. By (1.63), this proves assertion i) in the proof of Lemma 1.18.

If $\lambda\in(0,+\infty)$, we can pass to the limit in (1.65) to deduce (1.50) □

Proof of Theorem 1.5. Thanks to (1.15), there exist a subsequence of ε, still denoted by ε, and $(u,p)\in H^1(\Omega)^3\times L^2(\Omega)$ such that (1.16) holds.

On the other hand, we observe that $(u_\varepsilon,p_\varepsilon)$ satisfies the variational equation

$$\left\{\begin{array}{l}\displaystyle\int_{\Omega_\varepsilon}Du_\varepsilon:Dv_\varepsilon\,dx-\int_{\Omega_\varepsilon}p_\varepsilon\operatorname{div}v_\varepsilon\,dx=\int_{\Omega_\varepsilon}f_\varepsilon v_\varepsilon\,dx+\int_{\Omega_\varepsilon}G_\varepsilon:Dv_\varepsilon\,dx+\int_{\Gamma_\varepsilon}g_\varepsilon v_\varepsilon\,dx'\\ \forall\,v_\varepsilon\in H^1(\Omega_\varepsilon)^3,\quad v_\varepsilon\nu=0\ \text{ on }\Gamma_\varepsilon,\quad v_\varepsilon=0\text{ on }\partial\Omega_\varepsilon\setminus\Gamma_\varepsilon.\end{array}\right. \tag{1.66}$$

The proof of Theorem 1.5 will be carried out using suitable test functions v_ε depending on the values of λ.

Step 1. We start with the most difficult case $\lambda\in(0,+\infty)$ (critical size), which we will carry out more in detail.

We consider $v\in C^1_c(\omega\times(-1,1))^3$, $\widehat{v}\in C^1_c(\omega;C^1_\sharp(\widehat{Q})^3)$, with $D_y\widehat{v}(x',y)=0$ a.e. in $\{y_3>M\}$, for some $M>0$, such that

$$\left\{\begin{array}{l}v(x',x_3)=v(x',0)\ \text{ if }x_3\leq 0\\ v_3(x',0)=0,\end{array}\right.\qquad\left\{\begin{array}{l}\widehat{v}(x',y',y_3)=\widehat{v}(x',y',0)\ \text{ if }y_3\leq 0\\ \widehat{v}_3(x',y',0)=-\lambda\nabla\Psi(y')v'(x',0).\end{array}\right. \tag{1.67}$$

Besides, we take $\zeta\in C^\infty(\mathbb{R})$ such that

$$\zeta(x_3)=1\text{ if }x_3<\frac{1}{3},\qquad \zeta(x_3)=0\text{ if }x_3>\frac{2}{3}, \tag{1.68}$$

and $R_\varepsilon>0$ such that

$$R_\varepsilon\to\infty,\quad R_\varepsilon\left[\left(\frac{\delta_\varepsilon}{\varepsilon^{\frac{3}{2}}}-\lambda\right)^2+\varepsilon\right]\to 0. \tag{1.69}$$

Then, we define $v_\varepsilon \in H^1(\Omega_\varepsilon)^3$ by

$$\begin{cases} v'_\varepsilon(x) = v'(x) + \sqrt{\varepsilon}\,\widehat{v}'(x', \dfrac{x}{\varepsilon})\zeta(x_3) \\ v_{\varepsilon,3}(x) = v_3(x) + \sqrt{\varepsilon}\Bigg[\widehat{v}_3(x', \dfrac{x}{\varepsilon})\zeta(x_3) + \\ \qquad\qquad +\zeta(\dfrac{x_3}{\varepsilon R_\varepsilon})\nabla\Psi(\dfrac{x'}{\varepsilon})\left(\left(\lambda - \dfrac{\delta_\varepsilon}{\varepsilon^{\frac{3}{2}}}\right)v'(x',0) - \dfrac{\delta_\varepsilon}{\varepsilon}\widehat{v}'(x', \dfrac{x'}{\varepsilon}, 0)\right)\Bigg]. \end{cases}$$

Since $v \in C^1_c(\omega \times (-1,1))^3$, $\widehat{v} \in C^1_c(\omega; C^1_\sharp(\widehat{Q})^3)$ and $\zeta(x_3) = 0$ if $x_3 > 2/3$, the sequence v_ε satisfies

$$v_\varepsilon = 0 \ \text{ on } \partial\Omega_\varepsilon \setminus \Gamma_\varepsilon. \tag{1.70}$$

Moreover, taking into account the properties (1.67) of v and $\widehat{v}$, it is not difficult to check that

$$v_\varepsilon \nu = 0 \ \text{ on } \Gamma_\varepsilon. \tag{1.71}$$

Properties (1.70) and (1.71) of v_ε allow us to take it as test function in (1.66). To simplify the calculus we will first estimate the derivative of v_ε.

Taking into account that $D_y\widehat{v} = 0$ a.e. in $\{y_3 > M\}$ and that $\zeta = 1$ a.e. on $\{x_3 < 1/3\}$, we have

$$Dv_\varepsilon(x) = Dv(x) + \frac{1}{\sqrt{\varepsilon}}D_y\widehat{v}(x', \frac{x}{\varepsilon}) + h_\varepsilon(x), \tag{1.72}$$

where, using that v, $\widehat{v}$ and ζ are bounded and have bounded derivatives, the function $h_\varepsilon \in C^0(\bar{\Omega}_\varepsilon)^{3\times 3}$ satisfies

$$|h_\varepsilon| \leq C\sqrt{\varepsilon} + C\left[\left(\frac{1}{\sqrt{\varepsilon}R_\varepsilon} + \frac{1}{\sqrt{\varepsilon}}\right)\left(\left|\frac{\delta_\varepsilon}{\varepsilon^{\frac{3}{2}}} - \lambda\right| + \frac{\delta_\varepsilon}{\varepsilon}\right) + \sqrt{\varepsilon}\left|\frac{\delta_\varepsilon}{\varepsilon^{\frac{3}{2}}} - \lambda\right| + \frac{\delta_\varepsilon}{\varepsilon^{\frac{3}{2}}}\right]\chi_{\{x_3 < \varepsilon R_\varepsilon\}},$$

a.e. in Ω_ε. Using that R_ε tends to infinity and that $\delta_\varepsilon/\varepsilon^{\frac{3}{2}}$ is bounded, we get

$$|h_\varepsilon| \leq C\sqrt{\varepsilon} + C\left[\frac{1}{\sqrt{\varepsilon}}\left|\frac{\delta_\varepsilon}{\varepsilon^{\frac{3}{2}}} - \lambda\right| + 1\right]\chi_{\{x_3 < \frac{2}{3}\varepsilon R_\varepsilon\}}, \quad \text{a.e. in } \Omega_\varepsilon.$$

Therefore, by (1.69), we have

$$\int_{\Omega_\varepsilon} |h_\varepsilon|^2 dx \leq O_\varepsilon + CR_\varepsilon\left[\left(\frac{\delta_\varepsilon}{\varepsilon^{\frac{3}{2}}} - \lambda\right)^2 + \varepsilon\right] = O_\varepsilon. \tag{1.73}$$

Taking v_ε as test function in (1.66) and using that $\|u_\varepsilon\|_{H^1(\Omega_\varepsilon)^3}$, $\|p_\varepsilon\|_{L^2(\Omega_\varepsilon)}$ are bounded, $\|v_\varepsilon - v\|_{C^0(\bar\Omega_\varepsilon)^3}$ tends to zero, (1.72) and (1.73), we get

$$\begin{aligned}&\int_{\Omega_\varepsilon} (Du_\varepsilon : Dv - p_\varepsilon \text{div } v)\, dx + \frac{1}{\sqrt{\varepsilon}} \int_{\Omega_\varepsilon} \Big(Du_\varepsilon : D_y\widehat{v}(x', \frac{x}{\varepsilon}) - p_\varepsilon \text{div}_y \widehat{v}(x', \frac{x}{\varepsilon}) \Big)\, dx \\ &= \int_{\Omega_\varepsilon} f_\varepsilon v\, dx + \int_{\Omega_\varepsilon} G_\varepsilon : \Big(Dv + \frac{1}{\sqrt{\varepsilon}} D_y \widehat{v}(x', \frac{x}{\varepsilon}) \Big)\, dx + \int_{\Gamma_\varepsilon} g_\varepsilon v\, d\sigma + O_\varepsilon.\end{aligned} \tag{1.74}$$

In this equality we use that

$$\int_{\Omega_\varepsilon \setminus \Omega} \Big| Dv + \frac{1}{\sqrt{\varepsilon}} D_y \widehat{v}(x', \frac{x}{\varepsilon}) \Big|^2 dx \leq \frac{C}{\varepsilon} |\Omega_\varepsilon \setminus \Omega| \leq C \frac{\delta_\varepsilon}{\varepsilon} \leq C\sqrt{\varepsilon},$$

and that (1.11) and $D_y\widehat{v} = 0$ a.e. in $\{y_3 > M\}$ imply

$$\frac{1}{\sqrt{\varepsilon}} \int_{\Omega_\varepsilon} |G_\varepsilon D_y \widehat{v}(x', \frac{x}{\varepsilon})|\, dx \leq \frac{C}{\sqrt{\varepsilon}} \left(\int_{\{x_3 < M\varepsilon\}} |G_\varepsilon|^2 dx \right)^{\frac{1}{2}} |\{x_3 < M\varepsilon\}|^{\frac{1}{2}} = O_\varepsilon.$$

Therefore, (1.74) can be written as

$$\begin{aligned}&\int_{\Omega} (Du_\varepsilon : Dv - p_\varepsilon \text{div } v)\, dx + \frac{1}{\sqrt{\varepsilon}} \int_{\Omega} \Big(Du_\varepsilon : D_y\widehat{v}(x', \frac{x}{\varepsilon}) - p_\varepsilon \text{div}_y \widehat{v}(x', \frac{x}{\varepsilon}) \Big)\, dx \\ &= \int_{\Omega} f_\varepsilon v\, dx + \int_{\Omega} G_\varepsilon : Dv\, dx + \int_{\Gamma_\varepsilon} g_\varepsilon v\, d\sigma + O_\varepsilon,\end{aligned}$$

which taking into account (1.12), (1.16) and (1.67) proves

$$\begin{aligned}&\int_{\Omega} (Du : Dv - p\, \text{div } v)\, dx + \frac{1}{\sqrt{\varepsilon}} \int_{\Omega} \Big(Du_\varepsilon : D_y\widehat{v}(x', \frac{x}{\varepsilon}) - p_\varepsilon \text{div}_y \widehat{v}(x', \frac{x}{\varepsilon}) \Big)\, dx \\ &= \int_{\Omega} f v\, dx + \int_{\Omega} G : Dv\, dx + \int_{\Gamma} g v\, d\sigma + O_\varepsilon.\end{aligned} \tag{1.75}$$

In order to estimate the second term in (1.75), we introduce the sequences $\widehat{u}_\varepsilon$, $\widehat{p}_\varepsilon$ respectively defined by (1.44) and (1.45). By (1.15) and Lemmas 1.17 and 1.18, we can assume that there exist $\widehat{p} \in L^2(\omega \times \widehat{Q})$ and $\widehat{u} \in L^2(\omega; \mathcal{V}^3)$ which satisfy (1.47), (1.50), (1.51) and (1.52),

$$\begin{aligned}&\frac{1}{\sqrt{\varepsilon}} \int_{\Omega} \Big(Du_\varepsilon : D_y\widehat{v}(x', \frac{x}{\varepsilon}) - p_\varepsilon \text{div}_y \widehat{v}(x', \frac{x}{\varepsilon}) \Big)\, dx \\ &= \int_\omega \int_{\widehat{Q}_M} \left(\frac{1}{\sqrt{\varepsilon}} D_y \widehat{u}_\varepsilon : D_y \widehat{v} - \sqrt{\varepsilon}\widehat{p}_\varepsilon \text{div}_y \widehat{v} \right) dy\, dx' \\ &= \int_{\omega \times \widehat{Q}} \big(D_y \widehat{u} : D_y \widehat{v} - \widehat{p}\, \text{div}_y \widehat{v} \big)\, dx' dy + O_\varepsilon.\end{aligned}$$

Substituting in (1.75) we set

$$\begin{aligned}&\int_\Omega (Du : Dv - p \operatorname{div} v)\, dx + \int_{\omega\times\widehat{Q}} \big(D_y\widehat{u} : D_y\widehat{v} - \widehat{p}\operatorname{div}_y\widehat{v}\big)\, dx'dy\\ &= \int_\Omega fv\, dx + \int_\Omega G : Dv\, dx + \int_\Gamma gv\, d\sigma,\end{aligned} \tag{1.76}$$

for every $v \in C^1_c(\omega\times(-1,1))^3$, $\widehat{v} \in C^1_c(\omega; C^1_\sharp(\widehat{Q})^3)$, with $D_y\widehat{v}(x',y) = 0$ a.e. in $\{y_3 > M\}$, for some $M > 0$, and such that (1.67) is satisfied. By density, this equality holds true for every $v \in H^1(\Omega)^3$, and every $\widehat{v} \in L^2(\omega; \mathcal{V}^3)$ such that

$$v = 0 \text{ on } \partial\Omega\setminus\Gamma,$$

$$v_3(x',0) = 0, \quad \widehat{v}_3(x',y',0) = -\lambda\nabla\Psi(y')v'(x',0), \quad \text{a.e. } (x',y') \in \omega\times Y'.$$

Let us now obtain an equation for u eliminating $\widehat{u}$ and $\widehat{p}$ in (1.76). For this purpose, we take $v = 0$ in (1.76). This proves that $(\widehat{u},\widehat{p})$ (extended by periodicity to $\omega\times\mathbb{R}^2\times(0,+\infty)$) is a solution of

$$\begin{cases} -\Delta_y\widehat{u} + \nabla_y\widehat{p} = 0 \ \text{ in } \mathbb{R}^2\times\mathbb{R}^+\\ \operatorname{div}_y\widehat{u} = 0 \ \text{ in } \mathbb{R}^2\times(0,+\infty)\\ (\widehat{u},\widehat{p}) \in \mathcal{V}^3\times L^2_\sharp(\widehat{Q})\\ \widehat{u}_3(x',y',0) = -\lambda\nabla\Psi(y')u'(x',0) \ \text{ on } \mathbb{R}^2\times\{0\}\\ -\partial_{y_3}\widehat{u}' = 0 \ \text{ on } \mathbb{R}^2\times\{0\},\end{cases} \tag{1.77}$$

a.e. in ω. Defining $(\widehat{\phi}^i, \widehat{q}^i)$, $i = 1,2$, by (1.20), we deduce by linearity and uniqueness

$$D_y\widehat{u}(x',y) = -\lambda(u_1(x',0)D_y\widehat{\phi}^1(y) + u_2(x',0)D_y\widehat{\phi}^2(y)) \ \text{ a.e. in } \mathbb{R}^2\times(0,+\infty), \tag{1.78}$$

$$\widehat{p}(x',y) = \lambda(u_1(x',0)\widehat{q}^1(y) + u_2(x',0)\widehat{q}^2(y)) \ \text{ a.e. in } \mathbb{R}^2\times(0,+\infty). \tag{1.79}$$

Now, for $v \in H^1(\Omega)^3$, with $v = 0$ on $\partial\Omega\setminus\Gamma$, $v_3 = 0$ on Γ, we take v and $\hat{v}(x',y) = -\lambda\big(v_1(x',0)\widehat{\phi}^1(y) + v_2(x',0)\widehat{\phi}^2(y)\big)$, as test functions in (1.76). Taking into account (1.78) we get

$$\begin{aligned}&\int_\Omega Du : Dv\, dx - \int_\Omega p \operatorname{div} v\, dx + \lambda^2\int_\Gamma Ru'v'\, dx'\\ &= \int_\Omega fv\, dx + \int_\Omega G : Dv\, dx + \int_\Gamma gv\, dx'.\end{aligned} \tag{1.80}$$

By the arbitrariness of v, this proves that (u,p) is a solution of (1.22).

Step 2. The case $\lambda = 0$.

As in Step 1, we consider $v \in C_c^1(\omega \times (-1,1))^3$, with $v(x', x_3) = v(x', 0)$ if $x_3 \leq 0$, $v_3 = 0$ on Γ. Then, for $\zeta \in C^\infty(\mathbb{R})$ which satisfies (1.68), we define $v_\varepsilon \in H^1(\omega \times (-1,1))^3$ by

$$\begin{cases} v'_\varepsilon(x) = v'(x) \\ v_{\varepsilon,3}(x) = v_3(x) - \dfrac{\delta_\varepsilon}{\varepsilon}\zeta(\dfrac{x_3}{\varepsilon})\nabla\Psi(\dfrac{x'}{\varepsilon})v'(x). \end{cases}$$

The sequence v_ε satisfies $v_\varepsilon \nu = 0$ on Γ_ε and $v_\varepsilon = 0$ on $\partial\Omega_\varepsilon \setminus \Gamma_\varepsilon$. We take v_ε as test function in (1.66). Using (1.11), (1.12), (1.16) and that $\lambda = 0$ implies

$$\int_{\Omega_\varepsilon} |D(v_\varepsilon - v)|^2\,dx \to 0, \quad \|v_\varepsilon - v\|_{L^\infty(\Omega_\varepsilon)^3} \to 0, \tag{1.81}$$

we can pass to the limit in (1.66) to get

$$\begin{cases} \displaystyle\int_\Omega Du : Dv\,dx - \int_\Omega p\,\mathrm{div}\,v\,dx = \int_\Omega fv\,dx + \int_\Omega G : Dv\,dx + \int_\Gamma gv\,dx' \\ \forall\, v \in H^1(\Omega)^3,\ v_3 = 0 \text{ on } \Gamma,\ v = 0 \text{ on } \partial\Omega \setminus \Gamma. \end{cases} \tag{1.82}$$

This is equivalent to (1.19).

Step 3. The case $\lambda = +\infty$.

We consider $v \in C_c^1(\omega \times (-1,1))^3$, with $v(x', x_3) = v(x', 0)$ if $x_3 \leq 0$, $v_3 = 0$ on Γ and such that

$$v'(x', 0)\nabla\Psi(y') = 0 \ \text{ a.e. } (x', y') \in \omega \times Y'.$$

Observe that the properties of v imply that $v\nu = 0$ on Γ_ε, $v = 0$ on $\partial\Omega_\varepsilon \setminus \Gamma_\varepsilon$. Taking $v_\varepsilon = v$ in (1.66), passing to the limit in ε and reasoning by density we get

$$\begin{cases} \displaystyle\int_\Omega Du : Dv\,dx - \int_\Omega p\,\mathrm{div}\,v\,dx = \int_\Omega fv\,dx + \int_\Omega G : Dv\,dx + \int_\Gamma gv\,dx' \\ \forall\, v \in H^1(\Omega)^3,\ v_3 = 0 \text{ on } \Gamma,\ v = 0 \text{ on } \partial\Omega \setminus \Gamma \\ v'(x', 0)\nabla\Psi(y') = 0 \ \text{ a.e. } (x', y') \in \omega \times Y'. \end{cases}$$

This is equivalent to (1.24). □

Proof of Proposition 1.6. To simplify the notation we just prove the result for the pair $(\widehat{\phi}^1, \widehat{q}^1)$ which will be just denoted by $(\widehat{\phi}, \widehat{q})$ (the case $i = 2$ is completely analogous).
Step 1. Existence of $\widehat{\phi}$.

The function $\widehat{\phi}$ can be defined equivalently as the solution of the variational problem

$$\begin{cases} \widehat{\phi} \in (\mathcal{V}/\mathbb{R})^2 \times \mathcal{V}, \quad \widehat{\phi}_3 = \partial_1 \Psi \text{ on } Y' \times \{0\}, \quad \operatorname{div} \widehat{\phi} = 0 \text{ in } \widehat{Q} \\ \displaystyle\int_{\widehat{Q}} D\widehat{\phi} : D\widehat{v}\, dy = 0 \\ \forall\, \widehat{v} \in (\mathcal{V}/\mathbb{R})^2 \times \mathcal{V}, \quad \widehat{v}_3 = 0 \text{ on } Y' \times \{0\}, \quad \operatorname{div} \widehat{v} = 0 \text{ in } \widehat{Q}, \end{cases} \tag{1.83}$$

and then its existence and uniqueness follows from the Lax-Milgram theorem.

Step 2. Extension of $\widehat{\phi}$ to $\mathbb{R}^3$ and existence of the pressure $\widehat{q}$.

For $\zeta \in C^\infty((-\infty, 0])$ with $\zeta(0) = 1$, $\frac{d\zeta}{dt}(0) = \frac{d^2\zeta}{dt^2}(0) = 0$ and $\zeta(t) = 0$ if $t < -1$, we extend $\widehat{\phi}$ to $Y' \times \mathbb{R}$ by

$$\begin{cases} \widehat{\phi}_1(y) = \widehat{\phi}_1(y', -y_3) - 2\dfrac{d\zeta}{dy_3}(y_3)\Psi(y') \\ \widehat{\phi}_2(y) = \widehat{\phi}_2(y', -y_3) \\ \widehat{\phi}_3(y) = -\widehat{\phi}_3(y', -y_3) + 2\zeta(y_3)\partial_1 \Psi(y'), \end{cases}$$

a.e. in $Y' \times (-\infty, 0)$. Then, denoting by $\mathcal{W}$ the space of functions $\widehat{w} : \mathbb{R}^3 \to \mathbb{R}$ such that

$$\widehat{w} \in H^1_\sharp(Y' \times (-M, M)),\ \forall\, M > 0, \qquad \nabla \widehat{w} \in L^2_\sharp(\widehat{Y}' \times \mathbb{R})^3,$$

we have that $\widehat{\phi}$ satisfies

$$\begin{cases} \widehat{\phi} \in \mathcal{W}^3, \quad \operatorname{div} \widehat{\phi} = 0 \text{ in } Y' \times \mathbb{R} \\ \displaystyle\int_{\widehat{Q}} (D\widehat{\phi} + H) : D\widehat{w}\, dy = 0 \\ \forall\, \widehat{w} \in \mathcal{W}^3, \quad \operatorname{div} \widehat{w} = 0 \text{ in } Y' \times \mathbb{R}, \end{cases} \tag{1.84}$$

where H is a matrix function defined by zero in $Y' \times ((-\infty, -1) \cup (0, +\infty))$ and by

$$H(y) = 2 \begin{pmatrix} \dfrac{d\zeta}{dy_3}(y_3)\nabla\Psi(y') & \dfrac{d^2\zeta}{dy_3^2}(y_3)\Psi(y') \\ 0 & 0 \\ -\zeta(y_3)\nabla\partial_1\Psi(y') & -\dfrac{d\zeta}{dt}(y_3)\partial_1\Psi(y') \end{pmatrix}, \quad \forall\, y \in Y' \times (-1, 0).$$

This implies (reasoning similarly to Lemma A.1 in [11], see also [15]) that for every $\widehat{\varphi} \in H^1(\mathbb{R}^3)^3$ with compact support and such that $\operatorname{div} \widehat{\varphi} = 0$ in $\mathbb{R}^3$, one has

$$\int_{\mathbb{R}^3} (D\widehat{\phi} + H) : D\widehat{\varphi}\, dy = 0,$$

and so, there exists $\widehat{q}^* \in L^2_{loc}(\mathbb{R}^3)$ such that

$$\int_{\mathbb{R}^3} \widehat{q}^* \mathrm{div}\, \widehat{\varphi} = \int_{\mathbb{R}^3} (D\widehat{\phi} + H) : D\widehat{\varphi}\, dy, \tag{1.85}$$

for every $\widehat{\varphi} \in H^1(\mathbb{R}^3)^3$ with compact support.

Step 3. Let us prove that $\widehat{q}^*$ is periodic with respect to y' with period Y'.

Since $D\widehat{\phi} + H$ is periodic with respect to y' with period Y', equation (1.85) implies that for every $\widehat{\varphi} \in H^1(\mathbb{R}^3)^3$ with compact support we have

$$\int_{\mathbb{R}^3} (\widehat{q}^*(y_1 + 1, y_2, y_3) - \widehat{q}^*(y))\mathrm{div}\, \widehat{\varphi}(y)\, dy = 0.$$

Using that for every $\widehat{h} \in L^2(\mathbb{R}^3)$ with compact support and mean value zero there exists $\widehat{\varphi} \in H^1(\mathbb{R}^3)^3$ with compact support and div $\widehat{\varphi} = \widehat{h}$ in $\mathbb{R}^3$, we deduce that $\widehat{q}^*$ satisfies

$$\int_{\mathbb{R}^3} (\widehat{q}^*(y_1 + 1, y_2, y_3) - \widehat{q}^*(y))\widehat{h}(y)\, dy = 0,$$

for every $\widehat{h} \in L^2(\mathbb{R}^3)$ with compact support and mean value zero. This implies that there exists $c \in \mathbb{R}$ satisfying

$$\widehat{q}^*(y_1 + 1, y_2, y_3) = \widehat{q}^*(y) + c \quad \text{a.e. } y \in \mathbb{R}^3. \tag{1.86}$$

We define

$$\widehat{b}(y) = (\partial_2 \widehat{\phi}^1(y) + H_{1,2}(y), \partial_3 \widehat{\phi}^1(y) + H_{1,3}(y)), \quad \bar{b}(y_2, y_3) = \int_0^1 \widehat{b}(y)\, dy_1$$

and, for $n \in \mathbb{N}$, $\widehat{\rho}_n : \mathbb{R} \to \mathbb{R}$ by

$$\widehat{\rho}_n(s) = \begin{cases} s & \text{if } 0 \le s \le 1 \\ 1 & \text{if } 1 \le s \le n \\ n + 1 - s & \text{if } n \le s \le n + 1 \\ 0 & \text{if } s \le 0 \text{ or } s \ge n + 1. \end{cases}$$

Then, for $\widehat{\eta} \in C_c^\infty(\mathbb{R}^2)$, we take $\widehat{\varphi}(y) = (\widehat{\rho}_n(y_1)\widehat{\eta}(y_2, y_3), 0, 0)$ as test function in (1.85). Using the periodicity of $D\widehat{\phi} + H$ and (1.86), we get

$$-nc \int_{\mathbb{R}^2} \widehat{\eta}\, dy_2\, dy_3 = \int_{\mathbb{R}^2} \widehat{\eta} \left[\int_0^1 (\widehat{q}^*(y) - \widehat{q}^*(y_1 + n, y_2, y_3))\, dy_1 \right] dy_2\, dy_3 = \int_{\mathbb{R}^3} \widehat{q}^* \mathrm{div}\, \widehat{\varphi}\, dy$$
$$= \int_0^{n+1} \widehat{\rho}_n(y_1) \int_{\mathbb{R}^2} \hat{b} \nabla \widehat{\eta}\, dy_2\, dy_3 dy_1 = n \int_{\mathbb{R}^2} \bar{b} \nabla \widehat{\eta}\, dy_2\, dy_3,$$

for every $\widehat{\eta} \in C_c^\infty(\mathbb{R}^2)$. This implies $c = \text{div}\, \bar{b}$ in $\mathbb{R}^2$. Integrating this equality in $(-\frac{1}{2}, \frac{1}{2}) \times (t,t+1)$, with $t > 0$ and taking into account that $\bar{b} = (\bar{b}_2, \bar{b}_3)$ is periodic of period 1 with respect to y_2 we deduce

$$c = \int_{-\frac{1}{2}}^{\frac{1}{2}} (\bar{b}_3(y_2, t+1) - \bar{b}_3(y_2, t)) dy_2,$$

Integrating again for $t \in (s, s+1)$ with $s > 0$, this implies

$$|c| \leq \int_{(\frac{-1}{2},\frac{1}{2})\times(s,s+2)} |\bar{b}_3(y_2, y_3)|\, dy_2\, dy_3 \leq \sqrt{2} \left(\int_{(\frac{-1}{2},\frac{1}{2})\times(s,s+2)} |\bar{b}_3(y_2, y_3)|^2\, dy_2\, dy_3 \right)^{\frac{1}{2}},$$

for every $s > 0$. Since $\|\bar{b}_3(y_2, y_3)\|_{L^2((-1/2,1/2)\times\mathbb{R})} < +\infty$, we conclude that $c = 0$ and then by (1.86) it follows $\widehat{q}^*(y_1 + 1, y_2, y_3) = \widehat{q}^*(y)$, for a.e. $y \in \mathbb{R}^3$. Analogously, $\widehat{q}^*(y_1, y_2 + 1, y_3) = \widehat{q}^*(y)$, for a.e. $y \in \mathbb{R}^3$. Thus $\widehat{q}$ is periodic with respect to y' with period Y'.

Step 4. To finish the proof of Proposition 1.6, let us prove that we can choose a representative $\widehat{q}$ of $\widehat{q}^*$ (remark that $\widehat{q}^*$ is defined up to a constant) such that for every $r \geq 2$

$$\|D\widehat{\phi}\|_{L^r(Y'\times\mathbb{R})^{3\times3}} + \|\nabla\widehat{q}\|_{L^r(Y'\times\mathbb{R})} < +\infty. \tag{1.87}$$

This will imply that the pair $(\widehat{\phi}, \widehat{q})$ satisfies the thesis of Proposition 1.6 in $\mathbb{R}^2 \times (0, +\infty)$ (the uniqueness of $(\widehat{\phi}, \widehat{q})$ in $\big((\mathcal{V}/\mathbb{R})^2 \times \mathcal{V}\big) \times L^2_\sharp(\widehat{Q})$ is straightforward).

First we prove that for every $r \geq 2$ there exists $C_r > 0$ such that for any $n \in \mathbb{N}$,

$$\left\{ \begin{array}{l} \|D\widehat{\phi}\|_{L^r(nY'\times(-n,n))^{3\times3}} + \|\widehat{q}^* - \bar{q}_n\|_{L^r(nY'\times(-n,n))} \\ \leq \dfrac{C_r}{n^{\frac{3(r-2)}{2r}}} \|D\widehat{\phi}\|_{L^2(2nY'\times(-2n,2n))^{3\times3}} + C_r \|H\|_{L^r(2nY'\times(-2n,2n))^{3\times3}}, \end{array} \right. \tag{1.88}$$

where we have denoted

$$\bar{q}_n = \frac{1}{2n^3} \int_{nY'\times(-n,n)} \widehat{q}^* dy.$$

For $n = 1$, the proof of (1.88) follows from Theorem 4.4, chapter 4, in [16], the general case follows using a dilatation which transforms $Y' \times (-1, 1)$ in $nY' \times (-n, n)$.

Using the periodicity with respect to y' of $\widehat{\phi}$, $\widehat{q}^*$ and H we can write (1.88) as

$$\left\{ \begin{array}{l} \|D\widehat{\phi}\|_{L^r(Y'\times(-n,n))^{3\times3}} + \|\widehat{q}^* - \bar{q}_n\|_{L^r(Y'\times(-n,n))} \\ \leq \dfrac{2C_r}{n^{\frac{(r-2)}{2r}}} \|D\widehat{\phi}\|_{L^2(Y'\times(-2n,2n))^{3\times3}} + 2C_r \|H\|_{L^r(Y'\times(-2n,2n))^{3\times3}}. \end{array} \right. \tag{1.89}$$

This inequality implies that, up to a subsequence, there exists the limit $\widehat{q}$ of $\widehat{q}^* - \bar{q}_n$. Passing to the limit in n in (1.89) we get (1.87). □

Proof of Theorem 1.10.

Step 1. Let us first prove (1.28).

The Rellich-Kondrachov theorem and (1.16) give that u_ε converges to u strongly in $L^2(\Omega)^3$. On the other hand, from (1.42) and Hölder's inequality we have

$$\lim_{\varepsilon\to 0}\int_{\Omega_\varepsilon\setminus\Omega}|u_\varepsilon|^2dx \leq \limsup_{\varepsilon\to 0}\left(\|u_\varepsilon\|^2_{L^6(\Omega_\varepsilon)^3}|\Omega_\varepsilon\setminus\Omega|^{\frac{2}{3}}\right) \leq \limsup_{\varepsilon\to 0} C\delta_\varepsilon^{\frac{2}{3}}\|u_\varepsilon\|^2_{H^1(\Omega_\varepsilon)^3} = 0. \quad (1.90)$$

This proves that (1.28)) holds (for any value of $\lambda \in [0,+\infty]$).

In Steps 2, 3 and 4, let us prove the corrector result for Du_ε and p_ε.

Step 2. We consider $v_\varepsilon \in H^1(\Omega_\varepsilon)^3$ such that

$$v_\varepsilon\nu = 0 \ \text{ on } \Gamma_\varepsilon, \quad \|v_\varepsilon\|_{H^1(\Omega_\varepsilon)^3} \leq C, \quad (1.91)$$

and such that there exists $v \in H^1(\Omega)^3$ with v_ε converging weakly to v in $H^1(\Omega)^3$. By Lemma 1.18, the third component v_3 of v vanishes on Γ, and if $\lambda = +\infty$ it also holds $v'(x',0)\nabla\Psi(y') = 0$ a.e. $(x',y') \in \omega\times Y'$.

Let us prove that for any $\varphi \in C^1_c(\omega\times(-1,1))$, we have

$$\begin{cases} \lim\limits_{\varepsilon\to 0}\left(\displaystyle\int_{\Omega_\varepsilon} Du_\varepsilon : Dv_\varepsilon\,\varphi\,dx - \int_{\Omega_\varepsilon} p_\varepsilon \operatorname{div} v_\varepsilon\,\varphi\,dx\right) \\ = \displaystyle\int_\Omega Du : Dv\,\varphi\,dx - \int_\Omega p\operatorname{div} v\,\varphi\,dx, \end{cases} \quad \text{if } \lambda = 0,+\infty, \quad (1.92)$$

$$\begin{cases} \lim\limits_{\varepsilon\to 0}\left(\displaystyle\int_{\Omega_\varepsilon} Du_\varepsilon : Dv_\varepsilon\,\varphi\,dx - \int_{\Omega_\varepsilon} p_\varepsilon \operatorname{div} v_\varepsilon\,\varphi\,dx\right) \\ = \displaystyle\int_\Omega Du : Dv\,\varphi\,dx - \int_\Omega p\operatorname{div} v\,\varphi\,dx + \lambda^2\int_\Gamma Ru'v'\varphi\,dx', \end{cases} \quad \text{if } \lambda \in (0,+\infty). \quad (1.93)$$

where R is defined by (1.21).

For this purpose, given $\varphi \in C^1_c(\omega\times(-1,1))$, we take $v_\varepsilon\varphi$ as test function in (1.9). This gives

$$\int_{\Omega_\varepsilon} Du_\varepsilon : D(v_\varepsilon\varphi)dx - \int_{\Omega_\varepsilon} p_\varepsilon \operatorname{div}(v_\varepsilon\varphi)dx = \int_{\Omega_\varepsilon} f_\varepsilon v_\varepsilon\varphi dx + \int_{\Omega_\varepsilon} G_\varepsilon : D(v_\varepsilon\varphi)dx + \int_{\Gamma_\varepsilon} g_\varepsilon v_\varepsilon\varphi d\sigma. \quad (1.94)$$

Let us pass to the limit in each term of this equality.

Using that f_ε converges weakly to f in $L^{\frac{6}{5}}(\Omega)^3$, $|f_\varepsilon|^6$ is equiintegrable, v_ε converges to v in measure in Ω and v_ε converges weakly to v in $L^6(\Omega)^3$, we have that $f_\varepsilon v_\varepsilon$ converges strongly to fv in $L^1(\Omega)$. Therefore

$$\int_\Omega f_\varepsilon v_\varepsilon \varphi\, dx = \int_\Omega f v\varphi\, dx + O_\varepsilon. \tag{1.95}$$

On the other hand, by (1.11), (1.42), we get

$$\int_{\Omega_\varepsilon\setminus\Omega} f_\varepsilon v_\varepsilon \varphi\, dx = O_\varepsilon. \tag{1.96}$$

Reasoning analogously with the second term of (1.94), thanks to the strong convergence of G_ε in $L^2(\Omega)^{3\times 3}$, we have

$$\int_{\Omega_\varepsilon} G_\varepsilon : D(v_\varepsilon\varphi)dx = \int_\Omega G : D(v\varphi)dx + O_\varepsilon. \tag{1.97}$$

For the las term in (1.94), we use

$$\int_{\Gamma_\varepsilon} g_\varepsilon v_\varepsilon \varphi\, d\sigma = \int_\omega (g_\varepsilon v_\varepsilon\varphi)\big(x', -\delta_\varepsilon\Psi(\frac{x'}{\varepsilon})\big)\sqrt{1+\left(\frac{\delta_\varepsilon}{\varepsilon}\right)^2\left|\nabla\Psi(\frac{x'}{\varepsilon})\right|^2}\, dx'.$$

The inequality

$$\left\{\begin{array}{l}\displaystyle\int_\omega \left|v_\varepsilon\big(x', -\delta_\varepsilon\Psi(\frac{x'}{\varepsilon})\big) - v_\varepsilon(x',0)\right|^2 dx' = \int_\omega\left|\int_{-\delta_\varepsilon\Psi(\frac{x'}{\varepsilon})}^0 \partial_3 v_\varepsilon(x',s)\, ds\right|^2 dx'\\[2ex] \displaystyle\leq C\delta_\varepsilon\int_{\Omega_\varepsilon}|\partial_3 v_\varepsilon|^2 dx = O_\varepsilon,\end{array}\right. \tag{1.98}$$

and the compact imbedding of $H^1(\Omega)$ into $L^2(\Gamma)$ give that $v_\varepsilon(x', -\delta_\varepsilon\Psi(\frac{x'}{\varepsilon}))$ converges strongly to $v(x',0)$ in $L^2(\omega)^3$.

Thus, the weak convergence of $g_\varepsilon(x', -\delta_\varepsilon\Psi(\frac{x'}{\varepsilon}))$ to $g(x')$ in $L^2(\omega)^3$, Ψ in $W^{2,\infty}_\sharp(Y')$ and $\delta_\varepsilon/\varepsilon$ tends to zero, imply

$$\int_{\Gamma_\varepsilon} g_\varepsilon v_\varepsilon\varphi\, d\sigma = \int_\Gamma g v\varphi\, d\sigma + O_\varepsilon. \tag{1.99}$$

As v_ε converges to v strongly in $L^2(\Omega)^3$ and p_ε converges to p weakly in $L^2(\Omega)$, we have

$$\int_\Omega p_\varepsilon v_\varepsilon \nabla\varphi\, dx = \int_\Omega p v\nabla\varphi\, dx + O_\varepsilon. \tag{1.100}$$

On the other hand, thanks to (1.42) and (1.15)

$$\left|\int_{\Omega_\varepsilon\setminus\Omega} p_\varepsilon v_\varepsilon \nabla\varphi\,dx\right| \leq \|p_\varepsilon\|_{L^2(\Omega_\varepsilon)}\|v_\varepsilon\|_{L^6(\Omega_\varepsilon)^3}\|\nabla\varphi\|_{L^3(\Omega_\varepsilon\setminus\Omega)^3} = O_\varepsilon,$$

which, together with (1.100), gives

$$\int_{\Omega_\varepsilon} p_\varepsilon \mathrm{div}\,(v_\varepsilon\varphi)\,dx = \int_{\Omega_\varepsilon} p_\varepsilon \mathrm{div}\,v_\varepsilon\,\varphi\,dx + \int_\Omega p\,v\nabla\varphi\,dx + O_\varepsilon. \tag{1.101}$$

Finally, from the equality

$$\int_{\Omega_\varepsilon} Du_\varepsilon : D(v_\varepsilon\varphi)\,dx = \int_{\Omega_\varepsilon} Du_\varepsilon : Dv_\varepsilon\,\varphi\,dx + \int_{\Omega_\varepsilon} Du_\varepsilon : \big(v_\varepsilon\otimes\nabla\varphi\big)\,dx,$$

the strong convergence of v_ε to v in $L^2(\Omega)^3$, the weak convergence of Du_ε to Du in $L^2(\Omega)^{3\times 3}$ and that by (1.42)

$$\left|\int_{\Omega_\varepsilon\setminus\Omega} Du_\varepsilon : \big(v_\varepsilon\otimes\nabla\varphi\big)\,dx\right| \leq \|Du_\varepsilon\|_{L^2(\Omega_\varepsilon)^{3\times 3}}\|v_\varepsilon\|_{L^6(\Omega_\varepsilon)^3}\|\nabla\varphi\|_{L^3(\Omega_\varepsilon\setminus\Omega)^3} = O_\varepsilon,$$

we derive

$$\int_{\Omega_\varepsilon} Du_\varepsilon : D(v_\varepsilon\varphi)\,dx = \int_{\Omega_\varepsilon} Du_\varepsilon : Dv_\varepsilon\,\varphi\,dx - \int_\Omega Du : (v\otimes\nabla\varphi)dx + O_\varepsilon. \tag{1.102}$$

By (1.94), (1.95), (1.96), (1.97), (1.99), (1.101) and (1.102), we have then proved

$$\lim_{\varepsilon\to 0}\left(\int_{\Omega_\varepsilon} Du_\varepsilon : Dv_\varepsilon\,\varphi\,dx - \int_{\Omega_\varepsilon} p_\varepsilon \mathrm{div}\,(v_\varepsilon)\varphi\,dx\right) = -\int_\Omega Du : (v\otimes\nabla\varphi)\,dx$$
$$+\int_\Omega p\,v\nabla\varphi\,dx + \int_\Omega fv\varphi\,dx + \int_\Omega G : D(v\varphi)\,dx + \int_\Gamma gv\varphi\,dx.$$

But using $v\varphi$ as test function in the equation satisfied by the pair (u,p) (Theorem 1.5) we have that the second member of the above equality is equal to

$$\begin{cases} \displaystyle\int_\Omega Du : Dv\,\varphi\,dx - \int_\Omega p\,\mathrm{div}\,v\,\varphi\,dx & \text{if } \lambda = 0, +\infty \\ \displaystyle\int_\Omega Du : Dv\,\varphi\,dx - \int_\Omega p\,\mathrm{div}\,v\,\varphi\,dx + \lambda^2\int_\Gamma Ru'v'\varphi\,dx' & \text{if } \lambda\in(0,+\infty). \end{cases}$$

This proves (1.92) and (1.93).

Step 3. Let us prove (1.29), (1.33).

Using $v_\varepsilon = u_\varepsilon$ in Step 2 and taking into account that $\operatorname{div} u_\varepsilon = 0$ in Ω_ε, equalities (1.92) and (1.93) give for every $\varphi \in C_c^1(\omega \times (-1,1))$, $\varphi \geq 0$ in $\omega \times (-1,1)$,

$$\lim_{\varepsilon\to 0} \int_{\Omega_\varepsilon} |Du_\varepsilon|^2 \varphi \, dx = \int_\Omega |Du|^2 \varphi \, dx, \quad \text{ if } \lambda = 0, +\infty, \tag{1.103}$$

$$\lim_{\varepsilon\to 0} \int_{\Omega_\varepsilon} |Du_\varepsilon|^2 \varphi \, dx = \int_\Omega |Du|^2 \varphi \, dx + \lambda^2 \int_\Gamma Ru'u'\varphi \, dx', \quad \text{ if } \lambda \in (0, +\infty). \tag{1.104}$$

Since u_ε converges weakly to u in $H^1(\Omega)^3$, equality (1.103) proves (1.29).

In order to prove (1.33), we take $r_\varepsilon > 0$ such that

$$\lim_{\varepsilon\to 0} r_\varepsilon = 0, \qquad \lim_{\varepsilon\to 0} \frac{r_\varepsilon}{\varepsilon} = +\infty, \qquad \lim_{\varepsilon\to 0} \frac{r_\varepsilon}{\varepsilon} \int_{\{x_3<\varepsilon\}} |Du|^2 dx = 0. \tag{1.105}$$

Then we decompose

$$\int_{\Omega_\varepsilon} |Du_\varepsilon|^2 \varphi \, dx = \int_{\Omega_\varepsilon \setminus \Omega} |Du_\varepsilon|^2 \varphi \, dx + \int_{\{x_3 > r_\varepsilon\}} |Du_\varepsilon|^2 \varphi \, dx + \int_{\{0<x_3<r_\varepsilon\}} |Du_\varepsilon|^2 \varphi \, dx. \tag{1.106}$$

Let us estimate each term in the right hand side of (1.106).

Clearly

$$\liminf_{\varepsilon\to 0} \int_{\Omega_\varepsilon \setminus \Omega} |Du_\varepsilon|^2 \varphi \, dx \geq 0. \tag{1.107}$$

For the second term in the right side of (1.107), we use that for every $\rho > 0$ the weak convergence of Du_ε to Du in $L^2(\omega \times (\rho, 1))^{3\times 3}$ gives

$$\liminf_{\varepsilon\to 0} \int_{\{x_3>r_\varepsilon\}} |Du_\varepsilon|^2 \varphi \, dx \geq \liminf_{\varepsilon\to 0} \int_{\{x_3>\rho\}} |Du_\varepsilon|^2 \varphi \, dx \geq \int_{\{x_3>\rho\}} |Du|^2 \varphi \, dx.$$

So we have

$$\liminf_{\varepsilon\to 0} \int_{\{x_3>r_\varepsilon\}} |Du_\varepsilon|^2 \varphi \, dx \geq \sup_{\rho>0} \int_{\{x_3>\rho\}} |Du|^2 \varphi \, dx = \int_\Omega |Du|^2 \varphi \, dx. \tag{1.108}$$

For the third term on the right hand side of (1.106), we take $M > 0$ and $\varepsilon > 0$ small enough such that $M < r_\varepsilon/\varepsilon$. Defining $\widehat{u}_\varepsilon$ by (1.44) and using the change of variables (1.46) and the uniform continuity of φ, we get

$$\left\{ \begin{array}{l} \displaystyle\int_{\Omega\cap\{x_3<r_\varepsilon\}} |Du_\varepsilon|^2 \varphi \, dx = \int_{\omega\times\widehat{Q}_{\frac{r_\varepsilon}{\varepsilon}}} \Big|D_y\big(\frac{\widehat{u}_\varepsilon}{\sqrt{\varepsilon}}\big)\Big|^2 \varphi(x',0)\, dx'dy + O_\varepsilon \\ \displaystyle\geq \int_{\omega\times\widehat{Q}_M} \Big|D_y\big(\frac{\widehat{u}_\varepsilon}{\sqrt{\varepsilon}}\big)\Big|^2 \varphi(x',0)\, dx'dy + O_\varepsilon, \end{array} \right. \tag{1.109}$$

On the other hand, we saw in Step 1 in the proof of Theorem 1.5 that $\widehat{u}_\varepsilon/\sqrt{\varepsilon}$ converges weakly to $\widehat{u}$, defined by (1.31), in $L^2(\omega_\rho; H^1(\widehat{Q}_M)^3)$, for every $\rho > 0$. Therefore

$$
\begin{aligned}
\liminf_{\varepsilon\to 0} \int_{\{0<x_3<r_\varepsilon\}} |Du_\varepsilon|^2 \varphi\, dx &\geq \sup_{M>0} \int_{\omega\times\widehat{Q}_M} |D_y\widehat{u}|^2 \varphi(x',0)\, dx'dy \\
&= \int_{\omega\times\widehat{Q}} |D_y\widehat{u}|^2 \varphi(x',0)\, dx'dy = \lambda^2 \int_\Gamma Ru'u'\varphi\, dx'.
\end{aligned} \tag{1.110}
$$

By (1.104), statements (1.106), (1.107), (1.108), (1.109) and (1.110) imply

$$
\lim_{\varepsilon\to 0} \int_{\Omega_\varepsilon\setminus\Omega} |Du_\varepsilon|^2 \varphi\, dx = 0, \tag{1.111}
$$

$$
\lim_{\varepsilon\to 0} \int_{\Omega\cap\{x_3>r_\varepsilon\}} |Du_\varepsilon|^2 \varphi\, dx = \int_\Omega |Du|^2 \varphi\, dx \tag{1.112}
$$

and

$$
\begin{aligned}
\lim_{\varepsilon\to 0} \int_{\Omega\cap\{x_3<r_\varepsilon\}} |Du_\varepsilon|^2 \varphi\, dx &= \lim_{\varepsilon\to 0} \int_{\omega\times\widehat{Q}_{\frac{r_\varepsilon}{\varepsilon}}} \big|D_y(\frac{\widehat{u}_\varepsilon}{\sqrt{\varepsilon}})\big|^2 \varphi(x',0)\, dx'dy \\
&= \int_{\omega\times\widehat{Q}} |D_y\widehat{u}|^2 \varphi(x',0)\, dx'dy.
\end{aligned} \tag{1.113}
$$

From (1.112), (1.113) and the weak convergence of u_ε to u in $H^1(\Omega)^3$ and of $\frac{1}{\sqrt{\varepsilon}} D_y\widehat{u}_\varepsilon$ to $D_y\widehat{u}$ in $L^2(\omega\times\widehat{Q}_M)^{3\times 3}$, for every $M > 0$, we obtain

$$
\lim_{\varepsilon\to 0} \int_{\Omega\cap\{x_3>r_\varepsilon\}} |D(u_\varepsilon - u)|^2 \varphi\, dx = 0, \tag{1.114}
$$

$$
\lim_{\varepsilon\to 0} \int_{\omega\times\widehat{Q}_{\frac{r_\varepsilon}{\varepsilon}}} \left| D_y\left(\frac{\widehat{u}_\varepsilon}{\sqrt{\varepsilon}} - \widehat{u}\right)\right|^2 \varphi(x',0)\, dx'dy = 0. \tag{1.115}
$$

Therefore, taking $\rho > 0$ such that $\varphi(x) = 0$ if $x' \notin \omega_\rho$ and using that $\widehat{u}_\varepsilon(x',y)$ does not depend on x' in $C_\varepsilon^{k'} \times Y$, for every $k' \in I_{\rho,\varepsilon}$, we get

$$
\left\{
\begin{aligned}
&\int_{\{0<x_3<r_\varepsilon\}} \left| Du_\varepsilon - \frac{1}{\varepsilon^2}\int_{C_\varepsilon(x')} D_y\left(\frac{\widehat{u}}{\sqrt{\varepsilon}}\right)(z',\frac{x}{\varepsilon})d\,z'\right|^2 \varphi dx \\
&= \varepsilon^3 \sum_{k'\in I_{\rho,\varepsilon}} \int_{\widehat{Q}_{\frac{r_\varepsilon}{\varepsilon}}} \left| \frac{1}{\varepsilon^{\frac{5}{2}}} \int_{C_\varepsilon^{k'}} D_y\left(\frac{\widehat{u}_\varepsilon(z',y)}{\sqrt{\varepsilon}} - \widehat{u}(z',y)\right) dz'\right|^2 \varphi(x',0)\, dy + O_\varepsilon \\
&\leq \int_{\omega\times\widehat{Q}_{\frac{r_\varepsilon}{\varepsilon}}} \left| \frac{1}{\sqrt{\varepsilon}} D_y\widehat{u}_\varepsilon - D_y\widehat{u}\right|^2 \varphi(x',0)\, dz'dy + O_\varepsilon = O_\varepsilon.
\end{aligned}
\right. \tag{1.116}
$$

This gives a corrector result for Du_ε in $\Omega \times \{x_3 < r_\varepsilon\}$ (this type of correctors is usual when we apply the unfolding method, see e.g. [10], [13], [14]).

Let us improve (1.116) using the smoothness properties of $\widehat{u}$. By (1.31), Holder's inequality, (1.25) with $r = 4$, Lemma 1.19 below and (1.105) we have

$$\int_{\{x_3<r_\varepsilon\}} \left| D_y \left(\frac{\widehat{u}}{\sqrt{\varepsilon}} \right)(x', \frac{x}{\varepsilon}) - \frac{1}{\varepsilon^2} \int_{C_\varepsilon(x')} D_y \left(\frac{\widehat{u}}{\sqrt{\varepsilon}} \right)(z', \frac{x}{\varepsilon})\, d\, z' \right|^2 \varphi\, dx$$

$$\leq \frac{C}{\varepsilon} \sum_{i=1}^{2} \int_{\{x_3<r_\varepsilon\}} \left| u_i(x',0) - \frac{1}{\varepsilon^2} \int_{C_\varepsilon(x')} u_i(z',0)\, dz' \right|^2 \left| D_y \widehat{\phi}^i (\frac{x}{\varepsilon}) \right|^2 \varphi\, dx$$

$$\leq \frac{C}{\varepsilon} \sum_{i=1}^{2} \left(\int_{\{x_3<r_\varepsilon\}} \left| u_i(x',0) - \frac{1}{\varepsilon^2} \int_{C_\varepsilon(x')} u_i(z',0) dz' \right|^4 \varphi dx \right)^{\frac{1}{2}} \left(\int_{\{x_3<r_\varepsilon\}} |D_y \widehat{\phi}^i (\frac{x}{\varepsilon})|^4 \varphi dx \right)^{\frac{1}{2}}$$

$$\leq C \sqrt{\frac{r_\varepsilon}{\varepsilon}} \sum_{i=1}^{2} \left(\int_{\omega_\rho} \left| u_i(x',0) - \frac{1}{\varepsilon^2} \int_{C_\varepsilon(x')} u_i(z',0)\, dz' \right|^4 dx' \right)^{\frac{1}{2}} \left(\int_{\widehat{Q}} \left| D_y \widehat{\phi}^i \right|^4 dy \right)^{\frac{1}{2}}$$

$$\leq C \sqrt{\frac{r_\varepsilon}{\varepsilon}} \left(\sum_{k' \in I_{\rho,\varepsilon}} \int_{C_\varepsilon^{k'}} \left| u'(x',0) - \frac{1}{\varepsilon^2} \int_{C_\varepsilon^{k'}} u'(z',0)\, dz' \right|^4 dx' \right)^{\frac{1}{2}}$$

$$\leq C \sqrt{\frac{r_\varepsilon}{\varepsilon}} \left(\sum_{k' \in I_{\rho,\varepsilon}} \left(\int_{C_\varepsilon^{k'} \times (0,\varepsilon)} |Du'|^2\, dx' \right)^2 \right)^{\frac{1}{2}} \leq C \sqrt{\frac{r_\varepsilon}{\varepsilon}} \left(\int_{\{x_3<\varepsilon\}} |Du'|^2\, dx' \right)^2 = O_\varepsilon.$$

Thus, (1.116) implies

$$\int_{\Omega \cap \{x_3<r_\varepsilon\}} \left| Du_\varepsilon - \frac{1}{\sqrt{\varepsilon}} D_y \widehat{u}(x', \frac{x}{\varepsilon}) \right|^2 \varphi dx = 0. \tag{1.117}$$

By (1.111), (1.114), (1.117) and

$$\lim_{\varepsilon \to 0} \int_{\Omega \cap \{x_3<r_\varepsilon\}} |Du|^2 dx = \lim_{\varepsilon \to 0} \frac{1}{\varepsilon} \int_{\Omega \cap \{x_3>r_\varepsilon\}} \left| D_y \widehat{u}(x', \frac{x}{\varepsilon}) \right|^2 dx = 0,$$

(the second equality is immediate using the change of variables $y = x/\varepsilon$) we deduce (1.33).

Step 4. Let us now prove that (1.30) and (1.34) hold.

For every $\varepsilon > 0$, let $L_\varepsilon : L^2_0(\Omega_\varepsilon) \longrightarrow H^1_0(\Omega_\varepsilon)^3$ be the linear continuous operator defined in Proposition 1.14-*i*), and let us denote $v_\varepsilon = L_\varepsilon(p_\varepsilon)$. Then $\operatorname{div} v_\varepsilon = p_\varepsilon$ in Ω_ε, and thanks to (1.15) and Proposition 1.14-*i*), we have

$$\|v_\varepsilon\|_{H^1_0(\Omega_\varepsilon)^3} \leq \|L_\varepsilon\| \|p_\varepsilon\|_{L^2(\Omega_\varepsilon)} \leq C, \quad \forall \varepsilon > 0. \tag{1.118}$$

Thus, it is not difficult to prove that there exist a subsequence of ε, still denoted by ε, and $v \in H_0^1(\Omega)^3$, with $\operatorname{div} v = p$ in Ω, such that v_ε converges weakly in $H^1(\Omega)^3$ to v. Taking this sequence v_ε in Step 2, equalities (1.92) and (1.93) give for every $\varphi \in C_c^1(\omega \times (-1,1))$, $\varphi \geq 0$ in $\omega \times (-1,1)$

$$\begin{cases} \lim_{\varepsilon\to 0} \left(\int_{\Omega_\varepsilon} Du_\varepsilon : Dv_\varepsilon \, \varphi \, dx - \int_{\Omega_\varepsilon} |p_\varepsilon|^2 \, \varphi \, dx \right) \\ = \int_\Omega Du : Dv \, \varphi \, dx - \int_\Omega |p|^2 \, \varphi \, dx. \end{cases} \qquad \text{if } \lambda = 0, +\infty, \tag{1.119}$$

$$\begin{cases} \lim_{\varepsilon\to 0} \left(\int_{\Omega_\varepsilon} Du_\varepsilon : Dv_\varepsilon \, \varphi \, dx - \int_{\Omega_\varepsilon} |p_\varepsilon|^2 \, \varphi \, dx \right) \\ = \int_\Omega Du : Dv \, \varphi \, dx - \int_\Omega |p|^2 \, \varphi \, dx + \lambda^2 \int_\Gamma R u' v' \varphi \, dx'. \end{cases} \qquad \text{if } \lambda \in (0, +\infty). \tag{1.120}$$

If $\lambda = 0,\ +\infty$, (1.29), (1.118) and the weak convergence of v_ε to v in $H^1(\Omega)^3$, give

$$\lim_{\varepsilon\to 0} \int_{\Omega_\varepsilon} Du_\varepsilon : Dv_\varepsilon \, \varphi \, dx = \int_\Omega Du : Dv \, \varphi \, dx,$$

and then, by (1.119), we deduce

$$\lim_{\varepsilon\to 0} \int_{\Omega_\varepsilon} |p_\varepsilon|^2 \varphi \, dx = \int_\Omega |p|^2 \, \varphi \, dx. \tag{1.121}$$

Since p_ε converge weakly to p in $L^2(\Omega)$, equality (1.121) proves (1.30).

If $\lambda \in (0, +\infty)$, we apply Lemma 1.18 to v_ε which gives the existence of $\hat{v} \in L^2(\Omega; \mathcal{V}^3)$ such that, as $v = 0$ on Γ, satisfies

$$\hat{v}_3(x', y', 0) = -\lambda \nabla \Psi(y') v'(x'0) = 0 \quad \text{a.e. } (x', y') \in \omega \times Y', \tag{1.122}$$

and such that, up to a subsequence, the sequence $\widehat{v}_\varepsilon(x', y) = v_\varepsilon(\varepsilon\kappa(\frac{x'}{\varepsilon}) + \varepsilon y', \varepsilon y_3)$ a.e. $(x', y') \in \omega_\rho \times \widehat{Y}_\varepsilon$ satisfies

$$\frac{1}{\sqrt{\varepsilon}} \widehat{v}_\varepsilon \rightharpoonup \widehat{v} \text{ in } L^2(\omega_\rho; H^1(\widehat{Q}_M)^3), \quad \forall \rho, M > 0. \tag{1.123}$$

In particular, taking into account that

$$\frac{1}{\sqrt{\varepsilon}} \operatorname{div}_y \widehat{v}_\varepsilon = \sqrt{\varepsilon} \widehat{p}_\varepsilon \quad \text{in } \omega_\rho \times \widehat{Y}_\varepsilon,$$

we deduce

$$\mathrm{div}_y \widehat{v} = \widehat{p} \quad \text{in } \omega \times \hat{Q}. \tag{1.124}$$

Using (1.29), (1.118), the weak convergence of v_ε to v in $H^1(\Omega)^3$, the change of variables (1.46), the uniform continuity of φ and (1.123) we obtain

$$\begin{aligned}
&\lim_{\varepsilon\to 0}\int_{\Omega_\varepsilon} Du_\varepsilon : Dv_\varepsilon\, \varphi\, dx = \int_\Omega Du : Dv\, \varphi\, dx + \lim_{\varepsilon\to 0}\int_{\Omega_\varepsilon} \frac{1}{\sqrt{\varepsilon}} D_y\widehat{u}(x', \frac{x}{\varepsilon}) : Dv_\varepsilon\, \varphi\, dx \\
&= \int_\Omega Du : Dv\, \varphi\, dx + \int_{\omega\times\widehat{Q}} D_y\widehat{u} : D_y\widehat{v}\, \varphi(x',0)\, dx'dy,
\end{aligned} \tag{1.125}$$

but by taking $\widehat{v}$ as test function in (1.77), (1.122) and (1.124) , we deduce

$$\int_{\omega\times\widehat{Q}} D_y\widehat{u} : D_y\widehat{v}\, \varphi(x',0)\, dx'dy = \int_{\omega\times\widehat{Q}} |\widehat{p}|^2\, \varphi(x',0)\, dx'dy. \tag{1.126}$$

From (1.120), (1.125) and (1.126) we prove

$$\lim_{\varepsilon\to 0}\int_{\Omega_\varepsilon} |p_\varepsilon|^2\, \varphi\, dx = \int_\Omega |p|^2\, \varphi\, dx + \int_{\omega\times\widehat{Q}} |\widehat{p}|^2\, \varphi(x',0)\, dx'dy.$$

This equality is analogous to (1.104) (recall that $\int_\Gamma Ru'u'\, dx' = \int_{\omega\times\hat{Q}} |D_y\widehat{u}|^2\, dx'dy$). Therefore, reasoning as in Step 3, we deduce (1.34).

□

Lemma 1.19 *There exists $C > 0$, such that for every $t > 0$ and every $u \in H^1(tY' \times (0,t))$, we have*

$$\int_{tY'} |u(x',0) - \int_{tY'} u(z',0)\, dz'|^4\, dx \le C \left(\int_{tY'\times(0,t)} |\nabla u|\, dx\right)^2.$$

Proof. The result is well known for $t = 1$. The general case follows using a dilatation which transforms $Y' \times (0,1)$ in $tY' \times (0,t)$.

□

Proof of Theorem 1.12. Clearly, we can always assume $\mu = 1$. On the other hand, assumption (1.15) implies that up to a subsequence, there exist $u \in H^1(\Omega)^3$, $p \in L^2_0(\Omega)$ such that (1.16) is satisfied.

We define $f_\varepsilon \in L^{\frac{6}{5}}(\Omega_\varepsilon)^3$ by $f_\varepsilon = f - (u_\varepsilon \cdot \nabla)u_\varepsilon$, and $g_\varepsilon \in L^2(\Gamma_\varepsilon)^3$ as $g_\varepsilon = -\gamma u_\varepsilon$.

By (1.42) and (1.15), we have

$$\|(u_\varepsilon\cdot\nabla)u_\varepsilon\|_{L^{\frac{3}{2}}(\Omega_\varepsilon)^3}\leq \|u_\varepsilon\|_{L^6(\Omega_\varepsilon)^3}\|Du_\varepsilon\|_{L^2(\Omega_\varepsilon)^{3\times3}}\leq C\|Du_\varepsilon\|^2_{L^2(\Omega_\varepsilon)^{3\times3}}\leq C.$$

This implies that (1.11) (with $G_\varepsilon=0$) and (1.26) are satisfied. By the convergence in measure of u_ε, this also implies

$$f_\varepsilon \rightharpoonup f-(u\cdot\nabla)u \ \text{ in } L^{\frac{6}{5}}(\Omega)^3.$$

The inequality

$$|u_\varepsilon(x',0)-u_\varepsilon(x',-\delta_\varepsilon\Psi(\frac{x'}{\varepsilon}))|^2=\left|\int_{\delta_\varepsilon\Psi(\frac{x'}{\varepsilon}))}^0\partial_3u_\varepsilon(x',t)dt\right|^2\leq C\delta_\varepsilon\int_{\delta_\varepsilon\Psi(\frac{x'}{\varepsilon}))}^0|\partial_3u_\varepsilon(x',t)|^2dt$$

and the compactness embedding of $H^1(\Omega)$ into $L^2(\Gamma)$ proves that

$$g_\varepsilon(x',-\delta_\varepsilon\Psi(\frac{x'}{\varepsilon}))\to-\gamma u(x',0)\ \text{ in } L^2(\omega)^3.$$

So, since $(u_\varepsilon,p_\varepsilon)$ satisfies (1.35) we can apply Theorems 1.5 and 1.10 to conclude Theorem 1.12.

□

1.5 The case where lim $\delta_\varepsilon/\varepsilon>0$.

Although our main interest in the present paper is to study the asymptotic behavior of a viscous fluid satisfying slip conditions on a boundary defined by $x_3=-\delta_\varepsilon\Psi(\frac{x'}{\varepsilon})$ whith $\delta_\varepsilon/\varepsilon$ tending to zero, we give in this section a simple proof of the fact that if $\lim\frac{d_\varepsilon}{\varepsilon}\in(0,+\infty]$, then the main result established in [12] (for the case $\delta_\varepsilon=\varepsilon$) still holds true. This is given by the following Theorem.

Theorem 1.20 *We consider Ω_ε, Ω, Γ_ε and Γ defined as in Section 1.2. We assume $\Psi\in W^{1,\infty}_\sharp(Y')$ and*

$$\lim_{\varepsilon\to0}\delta_\varepsilon=0,\qquad \lim_{\varepsilon\to0}\frac{\delta_\varepsilon}{\varepsilon}\in(0,+\infty]. \tag{1.127}$$

Then, for every sequence $u_\varepsilon\in H^1(\Omega_\varepsilon)^3$, such that $\|u_\varepsilon\|_{H^1(\Omega_\varepsilon)^3}$ is bounded and satisfies $u_\varepsilon\nu=0$ on Γ_ε, we have that the weak limit $u=(u',u_3)$ of u_ε in $H^1(\Omega)^3$ (which exists at least for a subsequence) satisfies

$$u_3(x',0)=0,\quad u'(x',0)\nabla\Psi(y')=0,\ \textit{a.e. } (x',y')\in\omega\times Y'. \tag{1.128}$$

Proof. Let $\hat{Y}$ be defined by

$$\hat{Y} = \{(y', y_3) \in Y' \times \mathbb{R} : \ -\Psi(y') < y_3 < 1\}.$$

Similarly to (1.44), for $\rho > 0$ we define $\hat{u}_\varepsilon \in L^2(\omega_\rho; H^1(\hat{Y})^3)$ by

$$\hat{u}_\varepsilon(x', y) = u_\varepsilon\big(\varepsilon\kappa(\frac{x'}{\varepsilon}) + \varepsilon y', \delta_\varepsilon y_3\big) \ \text{ a.e. } (x', y') \in \omega_\rho \times \hat{Y},$$

where the function κ is defined in Section 1.2. Using

$$\int_{\omega_\rho \times \hat{Y}} \left(\frac{\delta_\varepsilon}{\varepsilon^2} |D_{y'} \hat{u}_\varepsilon|^2 + \frac{1}{\delta_\varepsilon} |\partial_{y_3} \hat{u}_\varepsilon|^2 \right) dx'dy \le \int_{\Omega_\varepsilon} |Du_\varepsilon|^2 dx$$

and

$$\int_{\omega_\rho \times Y'} |\hat{u}_\varepsilon(x', y', 0)|^2 dx'dy' \le \int_\omega |\hat{u}_\varepsilon(x'0)|^2 dx',$$

we have that $\hat{u}_\varepsilon$ is bounded in $L^2(\omega_\rho; H^1(\hat{Y})^3)$ and that $D_y \hat{u}_\varepsilon$ tends to zero in $L^2(\omega_\rho \times \hat{Y})^{3\times 3}$. Therefore, extracting a subsequence if necessary, we can assume that there exists a function $\widehat{u} \in L^2(\omega_\rho)^3$ such that

$$\hat{u}_\varepsilon \rightharpoonup \hat{u} \ \text{ in } L^2(\omega_\rho; H^1(\hat{Y})^3). \tag{1.129}$$

On the other hand, since the weak convergence of u_ε in $H^1(\Omega_\varepsilon)^3$ implies that $u_\varepsilon(\cdot, 0)$ converges strongly to $u(\cdot, 0)$ in $L^2(\omega)^3$, the two-scale limit of $u_\varepsilon(\cdot, 0)$ (see e.g. [1], [18]) coincides with $u(\cdot, 0)$. Since the limit given by the unfolding method coincides with the limit given by the two-scale convergence (see e.g. [14], [17]), this means that $\hat{u}_\varepsilon(x', y', 0) = u_\varepsilon(\varepsilon\kappa(\frac{x'}{\varepsilon}) + \varepsilon y', 0)$ converges weakly (in fact strongly) to $u(\cdot, 0)$ in $L^2(\omega_\rho \times Y')^3$. Using that the function $\hat{u}$ which appears in (1.129) does not depend on y', we then have

$$\hat{u}(x') = u(x', 0) \ \text{ a.e. } x' \in \omega. \tag{1.130}$$

Now, we remark that the equality $u_\varepsilon \nu = 0$ on Γ_ε can be written as

$$\frac{\delta_\varepsilon}{\varepsilon} \hat{u}'_\varepsilon(x', -\Psi(y'))\nabla\Psi(y') + \hat{u}_{\varepsilon,3}(x', -\Psi(y')) = 0 \ \text{ a.e. } (x', y') \in \omega_\rho \times Y'. \tag{1.131}$$

If $\frac{\delta_\varepsilon}{\varepsilon}$ converges to $\eta \in (0, +\infty)$, passing to the limit in this equality by (1.129) and (1.130), and taking into account the arbitrariness of ρ, we conclude

$$\eta u'(x', 0)\nabla\Psi(y') + u_3(x', 0) = 0 \ \text{ a.e. } x' \in \omega \times Y'. \tag{1.132}$$

Taking the integral in $y' \in Y'$ in this equality and using the periodicity of Ψ, we get the first equality in (1.128) and then (1.132) gives the second equality in (1.128).

If $\frac{\delta_\varepsilon}{\varepsilon}$ converges to $+\infty$, (1.131) implies that $u'(x', 0)\nabla\Psi(y') = \lim_{\varepsilon\to 0} \widehat{u}'_\varepsilon(x', -\Psi(y'))\nabla\Psi(y') = 0$ in $L^2(\omega \times Y')$ and then the second equality in (1.128). For the first one we reason as in Step 1 of the proof of Lemma 1.18.

□

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Chapter 2

Estimates for the asymptotic expansion of a viscous fluid satisfying Navier's law on a rugous boundary

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Abstract.

In a previous paper, we have studied the asymptotic behavior of a viscous fluid satisfying Navier's law on a periodic rugous boundary of period ε and amplitude δ_ε, with $\delta_\varepsilon/\varepsilon$ tending to zero. In the critical size, $\delta_\varepsilon \sim \varepsilon^{\frac{3}{2}}$, in order to obtain a strong approximation of the velocity and the pressure it is necessary to consider a boundary layer term in the corresponding ansatz. The purpose of the present paper is to estimate the approximation given by this ansatz.

2.1 Introduction

For a smooth open set $\omega \subset \mathbb{R}^2$, and a smooth function Ψ periodic of period $Y' = (0,1)^2$, we have studied in [15] (see also ([16]) for the case of a thin film) the asymptotic behavior of a viscous fluid in the domain $\Omega_\varepsilon \subset \mathbb{R}^3$ described by

$$\Omega_\varepsilon = \Big\{(x_1, x_2, x_3) :\ (x_1, x_2) \in \omega,\ -\delta_\varepsilon \Psi(\frac{x_1}{\varepsilon}, \frac{x_2}{\varepsilon}) < x_3 < 1\Big\},$$

where ε, δ_ε are two positive parameters such that ε, δ_ε and $\delta_\varepsilon/\varepsilon$ tend to zero. The boundary condition assumed on the slightly rugous boundary Γ_ε

$$\Gamma_\varepsilon = \Big\{(x_1, x_2, x_3) :\ (x_1, x_2) \in \omega,\ x_3 = -\delta_\varepsilon \Psi(\frac{x_1}{\varepsilon}, \frac{x_2}{\varepsilon})\Big\}$$

was not the usual adherence condition but the Navier law

$$u_\varepsilon \cdot \nu = 0 \text{ on } \Gamma_\varepsilon, \qquad \frac{\partial u_\varepsilon}{\partial \nu} \text{ parallel to } \nu \text{ on } \Gamma_\varepsilon,$$

where ν denotes the unitary outside normal vector to Ω_ε on Γ_ε and u_ε the velocity of the fluid. Depending on the limit λ of $\delta_\varepsilon/\varepsilon^{\frac{3}{2}}$, it was proved the existence of three different regimes in the behavior of the fluid.

If $\lambda = +\infty$, the fluid behaves as if we imposed Dirichlet conditions not only for the normal velocity on Γ_ε, but on the projection of u_ε on the linear space generated by the vectors

$$\big\{(\partial_{y_1}\Psi(y_1, y_2), \partial_{y_2}\Psi(y_1, y_2), 0) :\ (y_1, y_2) \in Y'\big\} \cup \big\{(0, 0, 1)\big\}.$$

In particular, if this space agrees with $\mathbb{R}^3$ (which always holds, except if Ψ only depends of one variable, i.e. $\Psi(y_1, y_2) = \Psi(y_1)$ or $\Psi(y_1, y_2) = \Psi(y_2)$), the fluid behaves as if we imposed the usual adherence condition $u_\varepsilon = 0$ on Γ_ε. This gives a mathematical explanation of why a viscous fluid adheres on the boundary. It can be due to the existence of micro-rugosities. The result extends the one obtained in [13] for $\delta_\varepsilon = \varepsilon$. See also [10] for a related result relative to a non-necessarily periodic boundary.

If $\lambda \in (0, +\infty)$, the boundary condition for the limit problem is

$$u_3 = 0 \text{ on } \{x_3 = 0\}, \qquad -\partial_3 u' + \lambda^2 R u' = 0 \text{ on } \{x_3 = 0\},$$

where $u = (u', u_3) \in \mathbb{R}^2 \times \mathbb{R}$ denotes the limit of u_ε and R is a nonnegative symmetric matrix of dimension 2×2. In this case the rugosity is not so large to imply the adherence condition in the limit but it makes to appear the friction term $\lambda^2 R u'$ which is similar to the *strange term* which appears in the homogenization of Dirichlet problems in varying domains (see e.g. [18]). A related result has been obtained in [11] for non-necessarily periodic boundaries.

If $\lambda = 0$ the rugosity is so small that it has not effect on the limit.

The above result is proved using the unfolding method ([5], [12], [14], [17], [19]), which is very related to the two-scale convergence method ([1], [20]).

The case $\lambda \in (0, +\infty)$ can be considered as the general one. The other cases can be obtained from this one passing to the limit in λ.

For $\lambda = 0, +\infty$ the velocity u_ε and the pressure p_ε converge strongly in the topologies of H^1 and L^2 respectively to the solutions u, p of the limit problem. However, for $\lambda \in (0, +\infty)$ the convergence is only weak. To obtain a strong convergence it is necessary to add a boundary term to the functions u and p. Namely, we have the approximation

$$\begin{cases} u_\varepsilon(x) \sim u(x) - \lambda\sqrt{\varepsilon}\Big(\widehat{\phi}^1(\frac{x}{\varepsilon})u_1(x) + \widehat{\phi}^2(\frac{x}{\varepsilon})u_2(x)\Big), \\ p_\varepsilon(x) \sim p(x) - \dfrac{\lambda}{\sqrt{\varepsilon}}\Big(\widehat{q}^1(\frac{x}{\varepsilon})u_1(x) + \widehat{q}^2(\frac{x}{\varepsilon})u_2(x)\Big), \end{cases} \tag{2.1}$$

where the pairs $(\widehat{\phi}^i, \widehat{q}^i)$, $i = 1, 2$, are the solutions of a Stokes problem in $\mathbb{R}^2 \times (0, +\infty)$ (see (2.3) below). Our purpose in the present paper is to obtain an error estimate for the differences of u_ε, p_ε and their respective correctors. We prove that they are of order $\sqrt{\varepsilon}$ in the topologies of H^1 and L^2 respectively. The exact result is given in Theorem 2.3.

To finish this introduction we refer to [2], [3], [4], [6], [7], [8] and [9] to other results relative to the behavior of viscous fluids in domains with rugous boundaries satisfying different boundary conditions of the ones imposed in the present work.

2.2 Error estimates

The present section is devoted to state the main result of the paper, Theorem 2.3, which estimates the difference between the left and right hand sides of (2.1).

We will decompose the points x of $\mathbb{R}^3$ as $x = (x', x_3)$. We also use the notation x' to refer to a point in $\mathbb{R}^2$.

For a bounded connected smooth open set $\omega \subset \mathbb{R}^2$, a function $\Psi \in W^{2,\infty}(\mathbb{R}^2)$ periodic of period $Y' = (0, 1)^2$ and two nonnegative numbers $\lambda, \varepsilon > 0$, we define

$$\Omega = \omega \times (0,1), \quad \Omega_\varepsilon = \Big\{x \in \mathbb{R}^3 : \ x' \in \omega, \ -\lambda\varepsilon^{\frac{3}{2}}\Psi(\frac{x'}{\varepsilon}) < x_3 < 1\Big\}$$

$$\Gamma = \omega \times \{0\}, \qquad \Gamma_\varepsilon = \Big\{x \in \mathbb{R}^3 : \ x' \in \omega, \ x_3 = -\lambda\varepsilon^{\frac{3}{2}}\Psi(\frac{x'}{\varepsilon})\Big\}.$$

For $f \in L^2(\mathbb{R}^3)^3$, let us consider the Stokes problem in Ω_ε

$$\begin{cases} -\Delta u_\varepsilon + \nabla p_\varepsilon = f \text{ in } \Omega_\varepsilon, \qquad \operatorname{div} u_\varepsilon = 0 \text{ in } \Omega_\varepsilon \\ u_\varepsilon \cdot \nu = 0 \text{ on } \Gamma_\varepsilon, \ \dfrac{\partial u_\varepsilon}{\partial \nu} \text{ parallel to } \nu \text{ on } \Gamma_\varepsilon, \ u_\varepsilon = 0 \text{ on } \partial\Omega_\varepsilon \setminus \Gamma_\varepsilon, \ \displaystyle\int_{\Omega_\varepsilon} p_\varepsilon \, dx = 0, \end{cases} \tag{2.2}$$

where ν denotes the outside unitary normal vector to Ω_ε on Γ_ε.

The asymptotic behavior of $(u_\varepsilon, p_\varepsilon)$ has been studied in [15]. We denote by $(\widehat{\phi}^i, \widehat{q}^i)$, $i = 1, 2$, the unique solution of the Stokes problem

$$\begin{cases} -\Delta\widehat{\phi}^i + \nabla\widehat{q}^i = 0 \;\text{ in } \mathbb{R}^2 \times \mathbb{R}^+, \qquad \operatorname{div}\widehat{\phi}^i = 0 \;\text{ in } \mathbb{R}^2 \times \mathbb{R}^+ \\ \widehat{\phi}^i(., y_3),\; \widehat{q}^i(., y_3) \;\text{ periodic, of period } Y', \text{ for a.e. } y_3 \in (0, +\infty) \\ D_y\widehat{\phi}^i \in L^2(Y' \times (0, +\infty))^{3\times 3}, \quad \widehat{q}^i \in L^2(Y' \times (0, +\infty)) \\ \widehat{\phi}^i_3 = \partial_i\Psi \;\text{ on } \mathbb{R}^2 \times \{0\}, \quad \partial_3\widehat{\phi}^i_1 = \partial_3\widehat{\phi}^i_2 = 0 \;\text{ on } \mathbb{R}^2 \times \{0\}, \quad \lim_{y_3\to\infty} \widehat{\phi}^i = 0. \end{cases} \tag{2.3}$$

Then, we define the matrix $R \in \mathbb{R}^{2\times 2}$ by

$$R_{ij} = \int_{\widehat{Q}} D_y\widehat{\phi}^i : D_y\widehat{\phi}^j \, dy, \quad \forall\, i, j \in \{1, 2\} \tag{2.4}$$

With these definitions, the following theorem is a consequence of the results proved in [15].

Theorem 2.1 *The solution $(u_\varepsilon, p_\varepsilon)$ of (2.2) converges weakly in $H^1(\Omega)^3 \times L^2(\Omega)$ to the unique solution (u, p) of the Stokes problem*

$$\begin{cases} -\Delta u + \nabla p = f \;\text{ in } \Omega, \quad \text{div}\, u = 0 \;\text{ in } \Omega \\ u_3 = 0 \;\text{ on } \Gamma, \quad -\partial_3 u' + \lambda^2 R u' = 0 \;\text{ on } \Gamma, \quad u = 0 \;\text{ on } \partial\Omega \setminus \Gamma, \quad \int_\Omega p\, dx = 0. \end{cases} \tag{2.5}$$

Moreover, taking

$$\tilde{u}_\varepsilon(x) = u(x) - \lambda\sqrt{\varepsilon}\Big(\widehat{\phi}^1(\frac{x}{\varepsilon})u_1(x) + \widehat{\phi}^2(\frac{x}{\varepsilon})u_2(x)\Big), \tag{2.6}$$

$$\tilde{p}_\varepsilon(x) = p(x) - \frac{\lambda}{\sqrt{\varepsilon}}\Big(\widehat{q}^1(\frac{x}{\varepsilon})u_1(x) + \widehat{q}^2(\frac{x}{\varepsilon})u_2(x)\Big), \tag{2.7}$$

the following corrector result holds

$$\lim_{\varepsilon\to 0} \big(\|u_\varepsilon\|_{H^1(\Omega_\varepsilon\setminus\Omega)^3} + \|p_\varepsilon\|_{L^2(\Omega_\varepsilon\setminus\Omega)} + \|u_\varepsilon - \tilde{u}_\varepsilon\|_{H^1(\Omega)^3} + \|p_\varepsilon - \tilde{p}_\varepsilon\|_{L^2(\Omega)}\big) = 0. \tag{2.8}$$

Remark 2.2 *In [15] the domain Ω_ε and the surface Γ_ε are respectively defined by*

$$\Omega_\varepsilon = \left\{x \in \mathbb{R}^3 :\; x' \in \omega,\; -\delta_\varepsilon\Psi(\frac{x'}{\varepsilon}) < x_3 < 1\right\}, \qquad \Gamma_\varepsilon = \left\{x \in \mathbb{R}^3 :\; x' \in \omega,\; x_3 = -\delta_\varepsilon\Psi(\frac{x'}{\varepsilon})\right\},$$

where δ_ε is an infinitesimal with respect to ε. As we explained in the introduction, the case considered here, $\delta_\varepsilon \sim \lambda\varepsilon^{\frac{3}{2}}$ is the most interesting one (critical size). The other cases can be obtained from this one taking λ converging to zero or plus infinity.

Our purpose is to estimate the differences $u_\varepsilon - \tilde u_\varepsilon$ and $p_\varepsilon - \tilde p_\varepsilon$. Indeed, instead of working with $\tilde u_\varepsilon$ and $\tilde p_\varepsilon$ which are only defined in Ω, and not in Ω_ε, let us consider the functions

$$u^*_\varepsilon(x) = \tilde u_\varepsilon(\eta_\varepsilon(x)), \qquad p^*_\varepsilon(x) = \tilde p_\varepsilon(\eta_\varepsilon(x)), \quad \text{a.e. } x \in \Omega_\varepsilon, \tag{2.9}$$

where $\eta_\varepsilon : \Omega_\varepsilon \to \Omega$ is given by

$$\eta_\varepsilon(x) = \left(x', \frac{x_3 + \lambda\varepsilon^{\frac{3}{2}}\Psi(\frac{x'}{\varepsilon})}{1 + \lambda\varepsilon^{\frac{3}{2}}\Psi(\frac{x'}{\varepsilon})} \right), \quad \forall\, x \in \Omega_\varepsilon.$$

Using these functions, we have the following theorem, which is the main result of the paper

Theorem 2.3 *We assume that the function u defined by (2.5) belongs to $H^s(\Omega)^3$, with $s > 3/2$. Then, there exists a constant $C > 0$ such that the solution $(u_\varepsilon, p_\varepsilon)$ of (2.2) and the functions u^*_ε, p^*_ε defined by (2.9) satisfy*

$$\|u_\varepsilon - u^*_\varepsilon\|_{H^1(\Omega_\varepsilon)^3} + \|p_\varepsilon - p^*_\varepsilon\|_{L^2(\Omega_\varepsilon)} \le C\sqrt{\varepsilon}. \tag{2.10}$$

As a corollary we get the following improvement of (2.8).

Corollary 2.4 *We assume that the function u defined by (2.5) belongs to $H^s(\Omega)^3$, with $s > 3/2$. Then, the solution $(u_\varepsilon, p_\varepsilon)$ of (2.2) and the functions $\tilde u_\varepsilon$, $\tilde p_\varepsilon$ defined by (2.6) and (2.7) satisfy*

$$\|u_\varepsilon\|_{H^1(\Omega_\varepsilon\setminus\Omega)^3} + \|p_\varepsilon\|_{L^2(\Omega_\varepsilon\setminus\Omega)} + \|u_\varepsilon - \tilde u_\varepsilon\|_{H^1(\Omega)^3} + \|p_\varepsilon - \tilde p_\varepsilon\|_{L^2(\Omega)} \le C\sqrt{\varepsilon}. \tag{2.11}$$

2.3 Proof of the error estimates.

Let us prove in the present section Theorem 2.3 and Corollary 2.4 estimating the difference between the solution $(u_\varepsilon, p_\varepsilon)$ of problem (2.2) and the asymptotic expansions defined by (2.9) and (2.6), (2.7) respectively.

Along this section, we denote by C a generic constant which does not depend on ε and can change from line to line.

To simplify the notation we will denote

$$\hat Q = Y' \times (0, +\infty)$$

where we recall that Y' refers to the unitary cube $(0,1)^2$. We will use the subindex $\sharp$ to denote periodicity with respect to Y'. For example, $L^2_\sharp(\hat Q)$ denotes the space of measurable functions h in $\mathbb{R}^2 \times (0, +\infty)$ such that for a.e. $y_3 \in (0, +\infty)$, $h(., y_3)$ is periodic of period Y' and satisfies $\|h\|_{L^2(\hat Q)} < +\infty$.

The following result giving some smoothness and decay at infinity properties of the solution $(\widehat{\phi}^i, \widehat{q}^i)$ of (2.3), is proved in [15].

Proposition 2.5 *For every $r \in [1, +\infty)$, one has*

$$\|D\widehat{\phi}^i\|_{L^r(\widehat{Q})^{3\times 3}} + \|\widehat{q}^i\|_{L^r(\widehat{Q})} < +\infty. \tag{2.12}$$

Moreover, $(\widehat{\phi}^i, \widehat{q}^i)$ belongs to $C^\infty_\sharp(\widehat{Q})^3 \times C^\infty_\sharp(\widehat{Q})$ and for every $\alpha \in \mathbb{N}^3$ and every $\delta > 0$, there exist two positive constants $C_{\delta,\alpha}$ and τ (the last one does not depend on δ or α) such that

$$|D^\alpha \widehat{\phi}^i(y)| + |D^\alpha \widehat{q}^i(y)| \le C_{\delta,\alpha} e^{-\tau y_3}, \quad \forall\, y \in \mathbb{R}^2 \times (\delta, +\infty). \tag{2.13}$$

Other interesting property, we will need later, of the functions $(\widehat{\phi}^i, \widehat{q}^i)$ is given by the following result.

Lemma 2.6 *Let $w \in H^s(\Omega)$ be with $s > 3/2$. Then, there exists $C > 0$ such that for every $i = 1, 2$, and every $v \in H^1(\Omega)^3$ with*

$$v = 0 \quad on\ \partial\Omega \setminus \Gamma, \qquad \lambda\sqrt{\varepsilon} v' \cdot \nabla_{y'} \Psi(\frac{x'}{\varepsilon}) + v_3 = 0 \quad on\ \Gamma, \tag{2.14}$$

one has

$$\left| \sqrt{\varepsilon} \int_\Omega D\left(w\widehat{\phi}^i(\frac{x}{\varepsilon})\right) : Dv\, dx - \frac{1}{\sqrt{\varepsilon}} \int_\Omega w\widehat{q}^i(\frac{x}{\varepsilon})\, div\, v\, dx + \lambda \sum_{j=1}^{2} \int_\Gamma R_{ij} w v_j\, dx' \right| \le C\sqrt{\varepsilon}\|v\|_{H^1(\Omega)^3}. \tag{2.15}$$

Proof. For $v \in H^1(\Omega)^3$ satisfying (2.14), we have

$$\begin{aligned} &\sqrt{\varepsilon} \int_\Omega D\left(w\widehat{\phi}^i(\frac{x}{\varepsilon})\right) : Dv\, dx - \frac{1}{\sqrt{\varepsilon}} \int_\Omega w\widehat{q}^i(\frac{x}{\varepsilon})\, \mathrm{div}\, v\, dx \\ &= \sqrt{\varepsilon} \int_\Omega \left(\widehat{\phi}^i(\frac{x}{\varepsilon}) \otimes \nabla w\right) : Dv\, dx - \frac{1}{\sqrt{\varepsilon}} \int_\Omega D_y\widehat{\phi}^i(\frac{x}{\varepsilon}) : (v \otimes \nabla w)\ dx \\ &+ \frac{1}{\sqrt{\varepsilon}} \int_\Omega \widehat{q}^i(\frac{x}{\varepsilon})\, (\nabla w \cdot v)\ dx + \frac{1}{\sqrt{\varepsilon}} \int_\Omega D_y\widehat{\phi}^i(\frac{x}{\varepsilon}) : D(wv)\, dx - \frac{1}{\sqrt{\varepsilon}} \int_\Omega \widehat{q}^i(\frac{x}{\varepsilon})\, \mathrm{div}\,(wv)\, dx. \end{aligned} \tag{2.16}$$

Let us estimate the right-hand side of this equality.

Since $\widehat{\phi}^i$ is in $L^\infty_\sharp(\hat{Q})^3$ and w belongs to $H^1(\Omega)$, the first term on the right-hand side of (2.16) satisfies

$$\left| \sqrt{\varepsilon} \int_\Omega \left(\widehat{\phi}^i(\frac{x}{\varepsilon}) \otimes \nabla w\right) : Dv\, dx \right| \le C\sqrt{\varepsilon}\|v\|_{H^1(\Omega)^3}. \tag{2.17}$$

For the second term on the right-hand side of (2.16) we use the decomposition

$$\begin{aligned} &\left| \frac{1}{\sqrt{\varepsilon}} \int_\Omega D_y\widehat{\phi}^i(\frac{x}{\varepsilon}) : (v \otimes \nabla w)\ dx \right| \\ &\le \frac{1}{\sqrt{\varepsilon}} \int_0^\varepsilon \int_\omega \left| D_y\widehat{\phi}^i(\frac{x}{\varepsilon})\right| |v||\nabla w|\, dx'\, dx_3 + \frac{1}{\sqrt{\varepsilon}} \int_\varepsilon^1 \int_\omega \left| D_y\widehat{\phi}^i(\frac{x}{\varepsilon})\right| |v||\nabla w|\, dx'\, dx_3. \end{aligned} \tag{2.18}$$

To estimate the two terms on the right hand side of this inequality, we use that due to w in $H^s(\Omega)$, with $s > 3/2$, there exists $\delta > 0$ such that for every $x_3 \in [0,1]$ the function $\nabla w(.,x_3) \in L^{2+\delta}(\omega)$ and

$$\int_\omega |\nabla w(x', x_3)|^{2+\delta}\, dx' \leq C, \quad \forall\, x_3 \in [0,1]. \tag{2.19}$$

Analogously, since v belongs to $H^1(\Omega)^3$, we have

$$\int_\omega |v(x', x_3)|^2\, dx' \leq C\|v\|^2_{H^1(\Omega)}, \quad \forall\, x_3 \in (0,1). \tag{2.20}$$

Therefore, using that $D_y\hat{\phi}^i$ belongs to $L_\sharp^{\frac{2(2+\delta)}{\delta}}(\hat{Q})^{3\times 3}$, we can estimate the first term on the right-hand side of (2.18) by

$$\begin{aligned}
&\frac{1}{\sqrt{\varepsilon}}\int_0^\varepsilon \int_\omega \left|D_y\widehat{\phi}^i(\frac{x}{\varepsilon})\right| |v||\nabla w|\, dx'\, dx_3 \\
&\leq \frac{1}{\sqrt{\varepsilon}}\int_0^\varepsilon \left(\int_\omega \left|D_y\widehat{\phi}^i(\frac{x}{\varepsilon})\right|^{\frac{2(2+\delta)}{\delta}} dx'\right)^{\frac{\delta}{2(2+\delta)}} \left(\int_\omega |v|^2 dx'\right)^{\frac{1}{2}} \left(\int_\omega |\nabla w|^{2+\delta} dx'\right)^{\frac{1}{2+\delta}} dx_3 \\
&\leq C\varepsilon^{\frac{1}{2+\delta}} \left(\int_0^\varepsilon \int_\omega \left|D_y\widehat{\phi}^i(\frac{x}{\varepsilon})\right|^{\frac{2(2+\delta)}{\delta}} dx' dx_3\right)^{\frac{\delta}{2(2+\delta)}} \|v\|_{H^1(\Omega)^3} \\
&= C\varepsilon^{\frac{1}{2+\delta}} \left(\varepsilon^3 \int_0^1 \int_{\frac{1}{\varepsilon}\omega} \left|D_y\widehat{\phi}^i\right|^{\frac{2(2+\delta)}{\delta}} dy' dy_3\right)^{\frac{\delta}{2(2+\delta)}} \|v\|_{H^1(\Omega)^3} \\
&\leq C\varepsilon^{\frac{1}{2+\delta}} \left(\varepsilon \int_{\hat{Q}} \left|D_y\widehat{\phi}^i\right|^{\frac{2(2+\delta)}{\delta}} dy\right)^{\frac{\delta}{2(2+\delta)}} \|v\|_{H^1(\Omega)^3} \leq C\sqrt{\varepsilon}\|v\|_{H^1(\Omega)^3}.
\end{aligned} \tag{2.21}$$

To estimate the second term on the right-hand side of (2.18) we use (2.19), (2.20) and the exponential decay at infinity of $D_y\widehat{\phi}^i$ given by (2.13). This gives

$$\frac{1}{\sqrt{\varepsilon}}\left|\int_\varepsilon^1 \int_\omega \left|D_y\widehat{\phi}^i(\frac{x}{\varepsilon})\right| |v||\nabla w|\, dx'\, dx_3\right| \leq \frac{C}{\sqrt{\varepsilon}}\int_\varepsilon^1 e^{-\frac{\tau x_3}{\varepsilon}} dx_3 \|v\|_{H^1(\Omega)} \leq C\sqrt{\varepsilon}\|v\|_{H^1(\Omega)^3},$$

which substituted in (2.18) and taking into account (2.21) proves

$$\left|\frac{1}{\sqrt{\varepsilon}}\int_\Omega D_y\widehat{\phi}^i(\frac{x}{\varepsilon}) : (v \otimes \nabla w)\ dx\right| \leq C\sqrt{\varepsilon}\|v\|_{H^1(\Omega)^3}. \tag{2.22}$$

A similar reasoning shows

$$\left| \frac{1}{\sqrt{\varepsilon}} \int_\Omega \widehat{q}^i(\frac{x}{\varepsilon})\,(\nabla w \cdot v)\;dx \right| \leq C\sqrt{\varepsilon}\|v\|_{H^1(\Omega)^3}. \tag{2.23}$$

It remains to estimate the fourth and fifth terms on the right-hand side of (2.16). For this purpose, we use that $(\widehat{\phi}^i, \widehat{q}^i)$ is a solution of (2.3). Taking into account that (2.14) and the boundary condition $\widehat{\phi}_3^i = \partial_i \Psi$ on $\mathbb{R}^2 \times \{0\}$ imply

$$wv + \lambda w\sqrt{\varepsilon}\left(\sum_{j=1}^2 v_j \widehat{\phi}^j(\frac{x}{\varepsilon})\right) = 0 \;\text{ on } \Omega \setminus \Gamma, \quad wv_3 + \lambda w\sqrt{\varepsilon}\left(\sum_{j=1}^2 v_j \widehat{\phi}_3^j(\frac{x}{\varepsilon})\right) = 0 \;\text{ on } \Gamma,$$

easily shows

$$\begin{aligned}
&\frac{1}{\sqrt{\varepsilon}}\int_\Omega D_y\widehat{\phi}^i(\frac{x}{\varepsilon}) : D(wv)\,dx - \frac{1}{\sqrt{\varepsilon}}\int_\Omega \widehat{q}^i(\frac{x}{\varepsilon})\mathrm{div}\,(wv)\,dx \\
&= -\lambda\sum_{j=1}^2\left(\int_\Omega D_y\widehat{\phi}^i(\frac{x}{\varepsilon}) : D\left(wv_j\widehat{\phi}^j(\frac{x}{\varepsilon})\right)\,dx - \int_\Omega \widehat{q}^i(\frac{x}{\varepsilon})\mathrm{div}\left(wv_j\widehat{\phi}^j(\frac{x}{\varepsilon})\right)dx\right) \\
&= -\lambda\sum_{j=1}^2\left(\int_\Omega D_y\widehat{\phi}^i(\frac{x}{\varepsilon}) : \left(\widehat{\phi}^j(\frac{x}{\varepsilon}) \otimes \nabla(wv_j)\right)dx - \int_\Omega \widehat{q}^i(\frac{x}{\varepsilon})\left(\nabla(wv_j)\cdot\widehat{\phi}^j(\frac{x}{\varepsilon})\right)dx\right) \\
&\quad -\frac{\lambda}{\varepsilon}\sum_{j=1}^2\int_\Omega D_y\widehat{\phi}^i(\frac{x}{\varepsilon}) : \left(wv_j D_y\widehat{\phi}^j(\frac{x}{\varepsilon})\right)dx.
\end{aligned} \tag{2.24}$$

Splitting the integral in Ω as the integral in $\omega\times(0,\varepsilon)$ plus the integral in $\omega\times(\varepsilon,1)$ (as we did in the estimation of the second term on the right-hand side of (2.16)), using that $w \in H^s(\Omega)$, with $s > 3/2$, that $\widehat{\phi}^1$, $\widehat{\phi}^2$ belong to $W_\sharp^{1,r}(\hat{Q})^3$, for every $r \geq 1$, $\widehat{q}^i$ belongs to $L_\sharp^r(\hat{Q})$, for every $r \geq 1$, and the exponential decay at infinity of $D_y\hat{\phi}^1$, $D_y\hat{\phi}^2$ and $\widehat{q}^i$ we easily show that

$$\begin{aligned}
&\left|\sum_{j=1}^2\left(\int_\Omega D_y\widehat{\phi}^i(\frac{x}{\varepsilon}) : \left(\widehat{\phi}^j(\frac{x}{\varepsilon})\otimes\nabla(wv_j)\right)dx - \int_\Omega \widehat{q}^i(\frac{x}{\varepsilon})\left(\nabla(wv_j)\cdot\widehat{\phi}^j(\frac{x}{\varepsilon})\right)dx\right)\right| \\
&\leq C\sqrt{\varepsilon}\|v\|_{H^1(\Omega)^3}.
\end{aligned} \tag{2.25}$$

It remains to estimate the last term in (2.24). Since $v = 0$ on $\partial\omega\times(0,1)$, we can assume

$wv_j = 0$ extended by zero to $\mathbb{R}^2 \times (0,1)$. Therefore, for a.e. $x_3 \in (0,1)$ one has

$$
\begin{aligned}
&\left(\int_\omega \left|(wv_j)(x', x_3) - \int_{Y'} (wv_j)(x' + \varepsilon y', 0)\, dy'\right|^2 dx'\right)^{\frac{1}{2}} \\
&\leq \left(\int_\omega \left|(wv_j)(x', x_3) - \int_{Y'} (wv_j)(x' + \varepsilon y', x_3)\, dy'\right|^2 dx'\right)^{\frac{1}{2}} \\
&\quad + \left(\int_\omega \left|\int_{Y'} \big((wv_j)(x' + \varepsilon y', x_3) - (wv_j)(x' + \varepsilon y', 0)\big)\, dy'\right|^2 dx'\right)^{\frac{1}{2}} \\
&\leq C\varepsilon \|wv\|_{H^1(\omega \times \{x_3\})^3} + C\sqrt{x_3}\|wv\|_{H^1(\Omega)^3}.
\end{aligned}
\tag{2.26}
$$

From this inequality and w in $H^s(\Omega)$, $s \geq 3/2$, it is easy to check

$$
\begin{aligned}
&\left|\frac{1}{\varepsilon}\int_\Omega D_y\widehat{\phi}^i(\frac{x}{\varepsilon}) : \Big(wv_j D_y\widehat{\phi}^j(\frac{x}{\varepsilon})\Big)\, dx - \int_\Gamma R_{ij} wv_j\, dx'\right| \\
&\leq \left|\frac{1}{\varepsilon}\int_\Omega \int_{Y'} D_y\widehat{\phi}^i(\frac{x}{\varepsilon}) : \Big((wv_j)(x' + \varepsilon y', 0) D_y\widehat{\phi}^j(\frac{x}{\varepsilon})\Big)\, dy'\, dx - \int_\Gamma R_{ij} wv_j\, dx'\right| \\
&+C\left(\left(\int_\Omega |D_y\widehat{\phi}^i(\frac{x}{\varepsilon})|^2 |D_y\widehat{\phi}^j(\frac{x}{\varepsilon})|^2 dx\right)^{\frac{1}{2}} + \int_0^1 \frac{\sqrt{x_3}}{\varepsilon}\left(\int_\omega |D_y\widehat{\phi}^i(\frac{x}{\varepsilon})||D_y\widehat{\phi}^j(\frac{x}{\varepsilon})|\, dx'\right)^{\frac{1}{2}} dx_3\right)\|v_j\|_{H^1(\Omega)}.
\end{aligned}
\tag{2.27}
$$

Using firstly the change of variables $z' = x' + \varepsilon y'$, then the change $y_3 = x_3/\varepsilon$ and the Y'-periodicity of $\hat{\phi}^i$, $i = 1, 2$, we get

$$
\begin{aligned}
&\frac{1}{\varepsilon}\int_\Omega \int_{Y'} D_y\widehat{\phi}^i(\frac{x}{\varepsilon}) : \Big((wv_j)(x' + \varepsilon y', 0) D_y\widehat{\phi}^j(\frac{x}{\varepsilon})\Big)\, dy'\, dx \\
&= \frac{1}{\varepsilon}\int_0^1 \int_{\mathbb{R}^2} \int_{Y'} D_y\widehat{\phi}^i(\frac{x}{\varepsilon}) : \Big((wv_j)(x' + \varepsilon y', 0) D_y\widehat{\phi}^j(\frac{x}{\varepsilon})\Big)\, dy'\, dx'\, dx_3 \\
&= \frac{1}{\varepsilon}\int_0^1 \int_{\mathbb{R}^2} \int_{Y'} D_y\widehat{\phi}^i(\frac{z'}{\varepsilon} - y', \frac{x_3}{\varepsilon}) : D_y\widehat{\phi}^j(\frac{z'}{\varepsilon} - y', \frac{x_3}{\varepsilon})\, dy'\, (wv_j)(z', 0)\, dz'\, dx_3 \\
&= \int_{\mathbb{R}^2} \int_{Y' \times \left(0, \frac{1}{\varepsilon}\right)} D_y\widehat{\phi}^i(y) : D_y\widehat{\phi}^j(y)\, dy\, (wv_j)(z', 0)\, dz'.
\end{aligned}
$$

which, taking into account the definition (2.4) of R, property (2.13) of $\hat{\phi}^i$, $i = 1, 2$, and

$w \in L^\infty(\Gamma)$, gives

$$\begin{aligned}&\left|\frac{1}{\varepsilon}\int_\Omega\int_{Y'} D_y\widehat{\phi}^i(\frac{x}{\varepsilon}) : \Big((wv_j)(x'+\varepsilon y',0)D_y\widehat{\phi}^j(\frac{x}{\varepsilon})\Big)\,dy'\,dx - \int_\Gamma R_{ij}wv_j\,dx'\right|\\ &\le \int_\omega\int_{Y'\times\left(\frac{1}{\varepsilon},+\infty\right)} |D_y\widehat{\phi}^i(y)|\,|D_y\widehat{\phi}^j(y)|\,dy\,|wv_j|(x',0)\,dx' \le Ce^{-\frac{2\tau}{\varepsilon}}\|v_j\|_{H^1(\Omega)}.\end{aligned} \tag{2.28}$$

On the other hand, by using the change of variables $y = x/\varepsilon$ and (2.12) with $r = 4$, we obtain

$$\int_\Omega |D_y\widehat{\phi}^i(\frac{x}{\varepsilon})|^2|D_y\widehat{\phi}^j(\frac{x}{\varepsilon})|^2dx \le C\varepsilon\int_{\hat{Q}} |D_y\widehat{\phi}^i(y)|^2|D_y\widehat{\phi}^j(y)|^2dy \le C\varepsilon. \tag{2.29}$$

Finally, splitting the integral in $(0,1)$ as the integral in $(0,\varepsilon)$ plus the integral in $(\varepsilon,1)$, and using again properties (2.12) and (2.13), we deduce

$$\begin{aligned}&\int_0^1 \frac{\sqrt{x_3}}{\varepsilon}\left(\int_\omega |D_y\widehat{\phi}^i(\frac{x}{\varepsilon})||D_y\widehat{\phi}^j(\frac{x}{\varepsilon})|\,dx'\right)^{\frac{1}{2}}dx_3\\ &\le \left(\int_0^\varepsilon \frac{x_3}{\varepsilon^2}dx_3\right)^{\frac{1}{2}}\left(\int_{\omega\times(0,\varepsilon)} |D_y\widehat{\phi}^i(\frac{x}{\varepsilon})|^2|D_y\widehat{\phi}^j(\frac{x}{\varepsilon})|^2dx\right)^{\frac{1}{2}} + C\int_\varepsilon^1 \frac{\sqrt{x_3}}{\varepsilon}e^{-2\tau\frac{x_3}{\varepsilon}}\,dx_3 \le C\sqrt{\varepsilon}.\end{aligned} \tag{2.30}$$

Thanks to (2.28), (2.29) and (2.30), estimate (2.27) reads

$$\left|\frac{1}{\varepsilon}\int_\Omega D_y\widehat{\phi}^i(\frac{x}{\varepsilon}) : \Big(wv_jD_y\widehat{\phi}^j(\frac{x}{\varepsilon})\Big)\,dx - \int_\Gamma R_{ij}wv_j\,dx'\right| \le C\sqrt{\varepsilon}\|v_j\|_{H^1(\Omega)}. \tag{2.31}$$

Using estimates (2.25) and (2.31) in (2.24) we then deduce

$$\begin{aligned}&\left|\frac{1}{\sqrt{\varepsilon}}\int_\Omega D_y\widehat{\phi}^i(\frac{x}{\varepsilon}) : D(wv)\,dx - \frac{1}{\sqrt{\varepsilon}}\int_\Omega \widehat{q}^i(\frac{x}{\varepsilon})\mathrm{div}\,(wv)\,dx + \lambda\sum_{j=1}^2\int_\Gamma R_{ij}wv_j\,dx'\right|\\ &\le C\sqrt{\varepsilon}\|v\|_{H^1(\Omega)^3}.\end{aligned} \tag{2.32}$$

By (2.16), (2.17), (2.22), (2.23), (2.32), we conclude (2.15). □

We are now in position to prove the main result of this paper.

Proof of Theorem 2.3. In order to estimate the differences $u_\varepsilon - u^*_\varepsilon$ and $p_\varepsilon - p^*_\varepsilon$, the idea is to show that $(u^*_\varepsilon, p^*_\varepsilon)$ is the solution of a Stokes problem similar to (2.2). This will be carried out in Steps 1 and 2. As a consequence we will conclude in Step 3 the proof of Theorem 2.3.

Step 1. Using that the function u^*_ε satisfies the following boundary conditions of $\partial\Omega_\varepsilon$

$$\begin{cases} u^*_\varepsilon = 0 \ \text{ on } \partial\Omega_\varepsilon \setminus \Gamma_\varepsilon \\ u^*_\varepsilon \cdot \nu = \dfrac{\lambda^2\varepsilon}{\sqrt{1+\lambda^2\varepsilon|\nabla_{y'}\Psi(\frac{x'}{\varepsilon})|^2}} \left(u_1(x',0)(\widehat{\phi}^1)'(\dfrac{x'}{\varepsilon},0) + u_2(x',0)(\widehat{\phi}^2)'(\dfrac{x'}{\varepsilon},0)\right) \cdot \nabla_{y'}\Psi(\dfrac{x'}{\varepsilon}) \ \text{ on } \Gamma_\varepsilon. \end{cases} \tag{2.33}$$

Therefore, taking $\rho \in C^\infty([0,1])$ such that $\rho(0) = 1$, $\rho(1) = 0$ and defining

$$h_\varepsilon(x) = \frac{\rho(x_3)\lambda^2\varepsilon}{\sqrt{1+\lambda^2\varepsilon|\nabla_{y'}\Psi(\frac{x'}{\varepsilon})|^2}} \left(u_1(x)(\widehat{\phi}^1)'(\frac{x}{\varepsilon}) + u_2(x)(\widehat{\phi}^2)'(\frac{x}{\varepsilon})\right) \cdot \nabla_{y'}\Psi(\frac{x'}{\varepsilon}),$$

we get that similarly to the function u_ε defined by (2.2), the sequence

$$u^{**}_\varepsilon = u^*_\varepsilon + (0, 0, (h_\varepsilon \circ \eta_\varepsilon))$$

satisfies

$$u^{**}_\varepsilon = 0 \ \text{ on } \partial\Omega_\varepsilon \setminus \Gamma_\varepsilon, \quad u^{**}_\varepsilon \cdot \nu = 0 \ \text{ on } \Gamma_\varepsilon. \tag{2.34}$$

Moreover, taking into account that

$$\|(\widehat{\phi}^i)'\|_{H^1_\sharp(\hat{Q})^2} \le C, \quad i = 1, 2,$$

we easily deduce that the difference between u^{**}_ε and u^*_ε can be estimated by

$$\|u^{**}_\varepsilon - u^*_\varepsilon\|_{H^1(\Omega_\varepsilon)^3} \le C\sqrt{\varepsilon}. \tag{2.35}$$

Step 2. Let us prove that for every $v \in H^1(\Omega_\varepsilon)^3$ which satisfies

$$v = 0 \text{ on } \partial\Omega_\varepsilon \setminus \Gamma_\varepsilon, \quad v \cdot \nu = 0 \text{ on } \Gamma_\varepsilon, \tag{2.36}$$

one has

$$\left|\int_{\Omega_\varepsilon} (Du^*_\varepsilon : Dv - p^*_\varepsilon \operatorname{div} v)\, dx - \int_\Omega f v dx\right| \le C\sqrt{\varepsilon}\|v\|_{H^1(\Omega_\varepsilon)^3}. \tag{2.37}$$

For this purpose, we define $\tilde{f} \in L^2(\Omega)^3$ and $\tilde{v} \in H^1(\Omega)^3$ by $\tilde{f} = f \circ \eta_\varepsilon^{-1}$ and $\tilde{v} = v \circ \eta_\varepsilon^{-1}$ respectively. Thanks to (2.36), the function $\tilde{v}$ satisfies (2.14) (with v replaced by $\tilde{v}$). Then, using the change of variables $z = \eta_\varepsilon(x)$, and that the jacobian matrix $D\eta_\varepsilon$ satisfies

$$\|D\eta_\varepsilon - I\|_{L^\infty(\Omega_\varepsilon)^{3\times 3}} \le C\sqrt{\varepsilon}, \tag{2.38}$$

we get

$$\left|\int_{\Omega_\varepsilon} \left(Du_\varepsilon^* : Dv - p_\varepsilon^* \mathrm{div}\, v\right)\, dx - \int_{\Omega_\varepsilon} f v\, dx\right|$$
$$\leq \left|\int_{\Omega} \left(D\tilde{u}_\varepsilon : D\tilde{v} - \tilde{p}_\varepsilon \mathrm{div}\, \tilde{v}\right)\, dz - \int_{\Omega_\varepsilon} f v\, dx\right| + C\sqrt{\varepsilon}\|v\|_{H^1(\Omega_\varepsilon)^3}.$$

Taking into account definitions (2.6), (2.7) of $\tilde{u}_\varepsilon$ and $\tilde{p}_\varepsilon$, the equation (2.5) satisfied by u and (2.15), we easily deduce

$$\left|\int_{\Omega_\varepsilon} \left(Du_\varepsilon^* : Dv - p_\varepsilon^* \mathrm{div}\, v\right)\, dx - \int_{\Omega_\varepsilon} f v\, dx\right| \leq \left|\int_{\Omega} f\tilde{v}\, dz - \int_{\Omega_\varepsilon} f v\, dx\right| + C\sqrt{\varepsilon}\|v\|_{H^1(\Omega_\varepsilon)^3}$$
$$\leq \int_{\Omega_\varepsilon\setminus\Omega} |fv|\, dx + \int_{\Omega} |f||v-\tilde{v}|\, dz + C\sqrt{\varepsilon}\|v\|_{H^1(\Omega_\varepsilon)^3} \leq C\sqrt{\varepsilon}\|v\|_{H^1(\Omega_\varepsilon)^3}.$$

This proves (2.37).

Step 3. By (2.2) and (2.37), we have that

$$\left|\int_{\Omega_\varepsilon} \left(D(u_\varepsilon - u_\varepsilon^*) : Dv - (p_\varepsilon - p_\varepsilon^*)\mathrm{div}\, v\right)\, dx\right| \leq C\sqrt{\varepsilon}\|v\|_{H^1(\Omega_\varepsilon)^3}, \tag{2.39}$$

for every $v \in H^1(\Omega_\varepsilon)^3$ which satisfies (2.36). This implies in particular that

$$\|\nabla(p_\varepsilon - p_\varepsilon^*)\|_{H^{-1}(\Omega_\varepsilon)} \leq C\left(\sqrt{\varepsilon} + \|u_\varepsilon - u_\varepsilon^*\|_{H^1(\Omega_\varepsilon)^3}\right), \tag{2.40}$$

which by Proposition 4.1 in [15] gives

$$\left\|p_\varepsilon - p_\varepsilon^* - \frac{1}{|\Omega_\varepsilon|}\int_{\Omega_\varepsilon}(p_\varepsilon - p_\varepsilon^*)dx\right\|_{L^2(\Omega_\varepsilon)} \leq C\left(\sqrt{\varepsilon} + \|u_\varepsilon - u_\varepsilon^*\|_{H^1(\Omega_\varepsilon)^3}\right). \tag{2.41}$$

Using that p_ε has zero integral in Ω_ε, p has zero integral in Ω, the functions $\widehat{q}^i$, $i = 1, 2$, are in $L^2_\sharp(\hat{Q})$, the function u is in $L^\infty(\Omega)^3$, the change of variables $z = \eta_\varepsilon(x)$ and (2.38), we also have

$$\left|\int_{\Omega_\varepsilon}(p_\varepsilon - p_\varepsilon^*)dx\right| \leq C\left(\sqrt{\varepsilon} + \sum_{i=1}^{2}\int_{\Omega}\left|\widehat{q}^i(\frac{x}{\varepsilon})\right| dx\right) \leq C\sqrt{\varepsilon},$$

and so, (2.41) reads as

$$\|p_\varepsilon - p_\varepsilon^*\|_{L^2(\Omega_\varepsilon)} \leq C\left(\sqrt{\varepsilon} + \|u_\varepsilon - u_\varepsilon^*\|_{H^1(\Omega_\varepsilon)^3}\right). \tag{2.42}$$

On the other hand, since u_ε^{**} satisfies (2.34), we can take $v = u_\varepsilon - u_\varepsilon^{**}$ in (2.39), which thanks to (2.35) gives

$$\begin{aligned}&\left|\int_{\Omega_\varepsilon} \left(D(u_\varepsilon - u_\varepsilon^*) : D(u_\varepsilon - u_\varepsilon^*) - (p_\varepsilon - p_\varepsilon^*)\mathrm{div}\,(u_\varepsilon - u_\varepsilon^*)\right)\, dx\right| \\ &\leq C\left(\sqrt{\varepsilon}\|u_\varepsilon - u_\varepsilon^*\|_{H^1(\Omega_\varepsilon)^3} + \varepsilon\right).\end{aligned} \tag{2.43}$$

Using that the functions u_ε, u, and $\widehat{\phi}^i$, $i = 1, 2$, have zero divergence, the change of variables $z = \eta_\varepsilon(x)$ and (2.38), we get

$$\begin{aligned}
&\int_{\Omega_\varepsilon} |\operatorname{div}(u_\varepsilon - u_\varepsilon^*)|^2\,dx \le C\int_\Omega |\operatorname{div}\tilde{u}_\varepsilon|^2\,dz + C\varepsilon\int_{\Omega_\varepsilon}|D\tilde{u}_\varepsilon|^2 dz \\
&\le C\sum_{i=1}^{2}\int_\Omega \left|\widehat{\phi}^i(\frac{z}{\varepsilon})\right|^2 |Du_i(z)|^2 dz + C\varepsilon.
\end{aligned} \tag{2.44}$$

Now, using for $i = 1, 2$ the decomposition

$$\int_\Omega \left|\widehat{\phi}^i(\frac{z}{\varepsilon})\right|^2 |Du_i(z)|^2 dz = \int_0^\varepsilon\int_\omega \left|\widehat{\phi}^i(\frac{z}{\varepsilon})\right|^2 |Du_i(z)|^2 dz' dz_3 + \int_\varepsilon^1\int_\omega \left|\widehat{\phi}^i(\frac{z}{\varepsilon})\right|^2 |Du_i(z)|^2 dz'\,dz_3,$$

taking into account that thanks to u in $H^s(\Omega)^3$, with $s > 3/2$, there exists $\delta > 0$ such that

$$\int_{\omega\times\{z_3\}} |Du|^{2+\delta} dz' \le C, \quad \forall\, z_3 \in (0, 1),$$

and the properties (2.12) and (2.13) of the functions $\widehat{\phi}^i$, $i = 1, 2$, we have

$$\int_\Omega \left|\widehat{\phi}^i(\frac{z}{\varepsilon})\right|^2 |Du_i(z)|^2 dz \le C\varepsilon,$$

which substituted in (2.44) shows

$$\|\operatorname{div}(u_\varepsilon - u_\varepsilon^*)\|_{L^2(\Omega_\varepsilon)} \le C\sqrt{\varepsilon}. \tag{2.45}$$

From (2.43), taking into account (2.42) and (2.45), we easily deduce

$$\|u_\varepsilon - u_\varepsilon^*\|_{H^1(\Omega_\varepsilon)^3} \le C\sqrt{\varepsilon}.$$

This inequality and (2.42) finally prove (2.10). □

Proof of Corollary (2.4). It is an easy consequence of (2.10) and the inequality

$$\|u_\varepsilon^*\|_{H^1(\Omega_\varepsilon\setminus\Omega)^3} + \|p_\varepsilon^*\|_{L^2(\Omega_\varepsilon\setminus\Omega)} + \|u_\varepsilon^* - \tilde{u}_\varepsilon\|_{H^1(\Omega)^3} + \|p_\varepsilon^* - \tilde{p}_\varepsilon\|_{L^2(\Omega)} \le C\sqrt{\varepsilon},$$

which is simple to check. □

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Chapter 3

A viscous fluid in a thin domain satisfying the slip condition on a slightly rough boundary

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Abstract.

We consider a viscous fluid of small height ε on a periodic rough bottom Γ_ε of period r_ε and amplitude δ_ε, $\delta_\varepsilon \ll r_\varepsilon \ll \varepsilon$, where we impose the slip boundary condition. When ε tends to zero we obtain a Reynolds system depending on the limit λ of $(\delta_\varepsilon\sqrt{\varepsilon})/(r_\varepsilon\sqrt{r_\varepsilon})$. If $\lambda = +\infty$, the fluid behaves as if we would impose the adherence condition on Γ_ε. This justifies why this is the usual boundary condition for viscous fluids. If $\lambda = 0$ the fluid behaves as if Γ_ε was plane. Finally, for $\lambda \in (0, +\infty)$ it behaves as if Γ_ε was flat but with a higher friction coefficient.

Résumé.

Un fluide visqueux dans un domaine de faible épaisseur qui vérifie la condition de glissement sur une frontière lègèrement rugueuse. On considère un fluide visqueux de petite hauteur ε sur un fond rugueux Γ_ε, périodique de période r_ε et amplitude δ_ε, $\delta_\varepsilon \ll r_\varepsilon \ll \varepsilon$, où on impose la condition de glissement. Quand ε converge vers zéro on

obtient un système de type Reynolds qui dépend de la limite λ de $(\delta_\varepsilon\sqrt{\varepsilon})/(r_\varepsilon\sqrt{r_\varepsilon})$. Si $\lambda = +\infty$, le fluide se comporte comme si on aurait imposé la condition d'adhérence sur Γ_ε. Ceci justifie qu'on impose d'habitude cette condition pour un fluide visqueux. Si $\lambda = 0$ le fluide se comporte comme si Γ_ε était plate. Enfin, pour $\lambda \in (0, +\infty)$, c'est comme si Γ_ε était plate, mais avec un coefficient de frottement plus élevé.

Version française abrégée

Pour un fluide visqueux dans un ouvert de $\mathbb{R}^3$ à frontière rugueuse, on sait que la condition de glissement et la condition d'adhérence sont asymptotiquement équivalentes. Ceci donne une justification mathématique de pourquoi on impose d'habitude la condition d'adhérence pour les fluides visqueux. L'équivalence entre la condition de glissement et la condition d'adhérence a été montré dans [10] dans le cas d'une frontière rugueuse de période ε et amplitude ε. Une extension à des frontières non-périodiques a été obtenue dans [8]. Dans [11], il a été considéré le cas d'une rugosité faible, plus exactement, la frontière est décrite par une fonction périodique de periode ε mais avec amplitude δ_ε, où δ_ε tend vers zéro. Alors, il a été montré que si $\delta_\varepsilon/\varepsilon^{3/2}$ tend vers l'infini, l'équivalence entre la condition de glissement et la condition d'adhérence est maintenue, mais si $\delta_\varepsilon/\varepsilon^{3/2}$ converge vers zéro le fluide se comporte comme si la frontière était plate. Dans le cas où $\delta_\varepsilon \sim \varepsilon^{3/2}$ la rugosité n'est pas assez grand pour impliquer la condition d'adhérence, mais elle est assez grande pour augmenter le coefficient de frottement. Un rèsultat général sur la forme de la limite du système de Navier-Stokes avec des conditions de glissement sur une frontière non necessairement périodique a été obtenue dans [7].

Dans cette Note, on généralise les résultats obtenus dans [11] au cas d'un domaine d'hauteur ε. Plus exactement, pour un ouvert borné Lipschitzien $\omega \subset \mathbb{R}^2$ et une fonction $\Psi \in W^{2,\infty}_{loc}(\mathbb{R}^2)$, périodique de période $Z' = (-1/2, 1/2)^2$, on définit Ω_ε par

$$\Omega_\varepsilon = \left\{ x = (x', x_3) \in \omega \times \mathbb{R} : \, -\delta_\varepsilon \Psi\left(\tfrac{x'}{r_\varepsilon}\right) < x_3 < \varepsilon \right\},$$

où les paramètres r_ε, δ_ε vérifient $\lim\limits_{\varepsilon\to 0} \dfrac{r_\varepsilon}{\varepsilon} = 0$, $\lim\limits_{\varepsilon\to 0} \dfrac{\delta_\varepsilon}{r_\varepsilon} = 0$. Alors, on considère un fluide satisfaisant le système de Stokes dans Ω_ε et la condition de Navier (ou glissement) $u_\varepsilon \cdot \nu = 0$, $\dfrac{\partial u_\varepsilon}{\partial \nu}$ parallel to ν sur la frontière rugueuse

$$\Gamma_\varepsilon = \left\{ x = (x', x_3) \in \omega \times \mathbb{R} : \, x_3 = -\delta_\varepsilon \Psi\big(\frac{x'}{r_\varepsilon}\big) \right\},$$

où $u_\varepsilon = (u'_\varepsilon, u_{\varepsilon,3})$ est la vitesse et ν la normal extérieure à Ω_ε sur Γ_ε. Pour simplifier on impose aussi la condition d'adhérence $u_\varepsilon = 0$ sur $\partial\Omega_\varepsilon \setminus \Gamma_\varepsilon$. Notre but est d'étudier le comportement asymptotique de ce système quand ε tend vers zéro. On obtient à la limite un système de type Reynolds qui dépend de $\lambda = \lim_{\varepsilon\to 0} \frac{\delta_\varepsilon\sqrt{\varepsilon}}{r_\varepsilon\sqrt{r_\varepsilon}}$. Grâce à ceci on déduit:

Si $\lambda = +\infty$, le fluide se comporte comme si on aurait supposé $\Gamma_\varepsilon = \{x_3 = 0\}$ avec la condition de frontière sur Γ_ε, $u_\varepsilon \in W^\perp \times \{0\}$, $\partial_3 u'_\varepsilon \in W$ avec $W = \{\nabla_{z'}\Psi(z') \in \mathbb{R}^2 : z' \in Z'\}$. En particulier, si W est de dimension 2, le fluide se comporte comme si on aurait imposé la condition d'adhérence dans Γ_ε.

Si $\lambda = 0$ on voit que la rugosité n'a pas d'effect à la limite.

Si $\lambda \in (0, +\infty)$, le fluide se comporte comme si on aurait supposé $\Gamma_\varepsilon = \{x_3 = 0\}$ avec la condition de frontière sur Γ_ε, $u_{\varepsilon,3} = 0$, $-\mu\partial_3 u'_\varepsilon + \lambda^2 R u'_\varepsilon = 0$, avec μ la viscosité du fluide et R une matrix symétrique carrée non-negative de dimension 2 qui est définie positive sur l'espace W. On observe que le nouveau terme $\lambda^2 R u'_\varepsilon$ c'est un terme de frottement. Cette condition de frontière peut être considerée comme la condition générale puisque quand λ tend vers zéro ou $+\infty$, elle donne les résultats antérieurs.

Ce résultat ressemble à celui qu'on a obtenu dans [11] pour un fluide d'hauteur fixe, mais la taille critique est différente de $\delta_\varepsilon \sim r_\varepsilon^{3/2}$ qui serait la taille correspondante à [11]. Ceci vient du fait que loin de la frontière rugueuse le comportement du fluid est différent. Dans notre cas on montre que la vitesse est d'ordre ε^2 et la pression est d'ordre 1 et ne dépend pas de la profondeur (dans une prémière approximation).

Pour finir on référence [1], [2], [4], [5], [6], [13], pour l'étude du comportement des fluides visqueux dans des domaines à frontière rugueuse, avec des conditions aux limites différents de celles qu'on a considéré dans ce papier.

3.1 Introduction

For a viscous fluid in an open set of $\mathbb{R}^3$ with a rugous boundary, it is known that if the normal velocity vanishes on the boundary (slip condition), then the fluid behaves as if the whole velocity vector vanishes on the boundary (adherence condition). This gives a mathematical explanation of why it is usual for a viscous fluid to impose the adherence condition. The equivalence between the slip and adherence conditions was proved in [10] for a periodic rough boundary of small period ε and amplitude ε. An extension to non-periodic boundaries was obtained in [8]. In [11] it was considered the case of a weak roughness, namely the boundary was described by a periodic function of small period ε and amplitude δ_ε, with $\delta_\varepsilon/\varepsilon$ converging to zero. It was proved that if $\delta_\varepsilon/\varepsilon^{3/2}$ tends to infinity, then the adherence and the slip conditions are still equivalent, while if $\delta_\varepsilon/\varepsilon^{3/2}$ tends to zero the fluid behaves as if the boundary was plane. In the critical case $\delta_\varepsilon \sim \varepsilon^{3/2}$ the roughness is not so large to imply the adherence condition but it is enough to increase the friction coefficient. A general result about the form of the limit equation for the Navier-Stokes system satisfying the slip condition on a (non-necessarily periodic) rough boundary has been obtained in [7].

Our aim in the present paper is to generalize the results in [11] to the case of a domain of small height ε. Namely, for a Lipschitz bounded open set $\omega \subset \mathbb{R}^2$ and a function Ψ in $W^{2,\infty}_{loc}(\mathbb{R}^2)$, periodic of period $Z' = (-1/2, 1/2)^2$, we define Ω_ε by

$$\Omega_\varepsilon = \left\{ x = (x', x_3) \in \omega \times \mathbb{R} : \, -\delta_\varepsilon \Psi\left(\frac{x'}{r_\varepsilon}\right) < x_3 < \varepsilon \right\}, \tag{3.1}$$

where the parameters r_ε, δ_ε are chosen non-negative and satisfying $\lim_{\varepsilon\to 0} \frac{r_\varepsilon}{\varepsilon} = 0$, $\lim_{\varepsilon\to 0} \frac{\delta_\varepsilon}{r_\varepsilon} = 0$. We consider a fluid satisfying the Stokes system in Ω_ε, the Navier (or slip condition) on the rough boundary

$$\Gamma_\varepsilon = \left\{ x = (x', x_3) \in \omega \times \mathbb{R} : \, x_3 = -\delta_\varepsilon \Psi\left(\frac{x'}{r_\varepsilon}\right) \right\} \tag{3.2}$$

and (to simplify) the adherence condition on the rest of the boundary $\partial\Omega_\varepsilon \setminus \Gamma_\varepsilon$. Our purpose is to study the asymptotic behavior of this system when ε tends to zero. We show that it depends on

$$\lambda = \lim_{\varepsilon\to 0} \frac{\delta_\varepsilon}{r_\varepsilon}\sqrt{\frac{\varepsilon}{r_\varepsilon}} \in [0, +\infty]. \tag{3.3}$$

If $\lambda = +\infty$ and the space $W = \{\nabla_{z'}\Psi(z') \in \mathbb{R}^2 : \, z' \in Z'\}$ agrees with $\mathbb{R}^2$, then the fluid behaves as if we impose the adherence condition on the whole $\partial\Omega_\varepsilon$.

If $\lambda = 0$, then the fluid behaves as if Γ_ε agrees with the plane boundary $\{x_3 = 0\}$.

If $\lambda \in (0, +\infty)$ the fluid behaves as if we had considered a plane boundary and added a friction coefficient to the Navier condition (see Theorem 3.1 and Remark 3.3).

This is analogous to the result proved in [11] for a fluid with fixed height, but the critical size is not $\delta_\varepsilon \sim r_\varepsilon^{3/2}$ which would be the expected size from [11]. This is due to the fact that far of the rugous boundary the behavior of the fluid is different from the corresponding one in [11]. Here one can show that the velocity is of order ε^2 while the pressure is of order 1 and does not depend on the depth (in a first approximation).

To finish this introduction we refer to [1], [2], [4], [5], [6], [13], for the study of viscous fluids in rugous domains satisfying different boundary conditions from the ones considered in the present paper.

3.2 Main results and some comments

Along this section, the points x of $\mathbb{R}^3$ are supposed to be decomposed as $x = (x', x_3)$ with $x' \in \mathbb{R}^2$, $x_3 \in \mathbb{R}$. We also use the notation x' to denote a generic vector of $\mathbb{R}^2$.

Given a bounded connected Lipschitz open set $\omega \subset \mathbb{R}^2$ and $\Psi \in W^{2,\infty}_{loc}(\mathbb{R}^2)$, periodic of period Z', we define Ω_ε by (3.1) and Γ_ε by (3.2). Then, for $f = (f', f_3) \in L^2(\omega)^3$, we consider the Stokes system in Ω_ε

$$\begin{cases} -\mu\Delta u_\varepsilon + \nabla p_\varepsilon = f \ \text{ in } \Omega_\varepsilon, & \operatorname{div} u_\varepsilon = 0 \ \text{ in } \Omega_\varepsilon, \\ u_\varepsilon = 0 \ \text{ on } \partial\Omega_\varepsilon \setminus \Gamma_\varepsilon, & u_\varepsilon \cdot \nu = 0 \ \text{ on } \Gamma_\varepsilon, \quad \frac{\partial u_\varepsilon}{\partial \nu} \ \text{ parallel to } \nu \ \text{ on } \Gamma_\varepsilon. \end{cases} \tag{3.4}$$

Here ν denotes the unitary outside normal vector to Ω_ε in Γ_ε and $\mu > 0$ corresponds to the viscosity of the fluid. It is well known that (3.4) has a unique solution $(u_\varepsilon, p_\varepsilon) \in H^1(\Omega_\varepsilon)^3 \times L^2_0(\Omega_\varepsilon)$ ($L^2_0(\Omega_\varepsilon)$ denotes the space of functions in $L^2(\Omega_\varepsilon)$ whose integral in Ω_ε is zero). Moreover, we can show the following estimates

$$\fint_{\Omega_\varepsilon} |u_\varepsilon|^2 dx \leq C\varepsilon^4, \qquad \fint_{\Omega_\varepsilon} |Du_\varepsilon|^2 dx \leq C\varepsilon^2, \qquad \fint_{\Omega_\varepsilon} |p_\varepsilon|^2 dx \leq C. \tag{3.5}$$

Our aim is to study the asymptotic behavior of u_ε and p_ε when ε tends to zero. For this purpose, as usual, we use a dilatation in the variable x_3 in order to have the functions defined in an open set of fixed height. Namely, we take $\Omega = \omega \times (0,1)$ and we define $\tilde{u}_\varepsilon \in H^1(\Omega)^3$, $\tilde{p}_\varepsilon \in L^2_0(\Omega)$ by

$$\tilde{u}_\varepsilon(y) = u_\varepsilon(y', \varepsilon y_3), \quad \tilde{p}_\varepsilon(y) = p_\varepsilon(y', \varepsilon y_3), \quad \text{a.e. } y \in \Omega. \tag{3.6}$$

Then, our problem is to describe the asymptotic behavior of these sequences $\tilde{u}_\varepsilon$, $\tilde{p}_\varepsilon$. This is given by the following theorem which is the main result of the present paper.

Theorem 3.1 *Let $(u_\varepsilon, p_\varepsilon) \in H^1(\Omega_\varepsilon)^3 \times L^2_0(\Omega_\varepsilon)$ be the solution of the Stokes system (3.4) and let $\tilde{u}_\varepsilon$, $\tilde{p}_\varepsilon$ be defined by (3.6). Then, there exist $v \in H^1(0,1;L^2(\omega))^2$, $w \in H^2(0,1;H^{-1}(\omega))$ and $p \in L^2_0(\Omega)$, where p does not depend on y_3, such that, up to a subsequence,*

$$\frac{\tilde{u}_\varepsilon}{\varepsilon} \rightharpoonup 0 \ \text{in } H^1(\Omega)^3, \quad \frac{\tilde{u}_\varepsilon}{\varepsilon^2} \rightharpoonup (v,0) \ \text{in } H^1(0,1;L^2(\omega))^3, \quad \frac{\tilde{u}_{\varepsilon,3}}{\varepsilon^3} \rightharpoonup w \ \text{in } H^2(0,1;H^{-1}(\omega)), \tag{3.7}$$

$$\tilde{p}_\varepsilon \rightharpoonup p \ \text{in } L^2(\Omega), \quad \frac{\partial_{y_3}\tilde{p}_\varepsilon}{\varepsilon} \rightharpoonup f_3 \ \text{in } H^{-1}(\Omega). \tag{3.8}$$

According to the value of λ defined by (3.3), the functions v, w and p are given by:

(i) If $\lambda = +\infty$, then denoting by $P_{W^\perp}$ the orthogonal projection from $\mathbb{R}^2$ to the orthogonal of the space $W = \{\nabla_{z'}\Psi(z') \in \mathbb{R}^2 : z' \in Z'\}$, we have that v and p are given by

$$v(y) = \frac{(y_3 - 1)}{2\mu}\Big(y_3 I + P_{W^\perp}\Big)\left(\nabla_{y'}p(y') - f'(y')\right), \qquad a.e. \ y \in \Omega,$$

$$-div_{y'}\left(\left(\frac{1}{3}I + P_{W^\perp}\right)(\nabla_{y'}p - f')\right) = 0 \ \text{in } \omega, \quad \left(\frac{1}{3}I + P_{W^\perp}\right)(\nabla_{y'}p - f') \cdot \nu = 0 \ \text{ on } \partial\omega.$$

Moreover, the distribution w is given by $\quad w(y) = -\displaystyle\int_0^{y_3} div_{y'} v(y', s)ds, \ \text{ in } \Omega. \quad (3.9)$

(ii) If $\lambda \in (0,+\infty)$, then defining $(\widehat{\phi}^i, \widehat{q}^i)$, $i = 1, 2$, as solutions of the Stokes systems

$$\begin{cases} -\mu\Delta_z\widehat{\phi}^i + \nabla_z\widehat{q}^i = 0 \quad \text{in } \mathbb{R}^2 \times (0,+\infty), \quad \operatorname{div}_z \widehat{\phi}^i = 0 \quad \text{in } \mathbb{R}^2 \times (0,+\infty), \\ \widehat{\phi}^i_3(z',0) + \partial_{z_i}\Psi(z') = 0, \ \partial_{z_3}(\widehat{\phi}^i)'(z',0) = 0, \quad \widehat{\phi}^i(.,z_3), \ \widehat{q}^i(.,z_3) \text{ periodic of period } Z', \\ D_z\widehat{\phi}^i \in L^2(Z' \times (0,+\infty))^{3\times 3}, \ \widehat{q}^i \in L^2(Z' \times (0,+\infty)), \end{cases}$$

and $R \in \mathbb{R}^{2\times 2}$ by $\quad R_{ij} = \mu \int_{Z'\times(0,+\infty)} D_z\widehat{\phi}^i : D_z\widehat{\phi}^j \, dz, \quad \forall\, i,j \in \{1,2\}$, *we have*

$$v(y) = \frac{(y_3-1)}{2\mu}\left(y_3 I + \left(I + \frac{\lambda^2}{\mu}R\right)^{-1}\right)(\nabla_{y'}p(y') - f'(y')), \qquad a.e.\ y \in \Omega,$$

where p satisfies
$$\begin{cases} -\operatorname{div}_{y'}\left(\left(\frac{1}{3}I + \left(I + \frac{\lambda^2}{\mu}R\right)^{-1}\right)(\nabla_{y'}p - f')\right) = 0 \quad \text{in } \omega, \\ \left(\frac{1}{3}I + \left(I + \frac{\lambda^2}{\mu}R\right)^{-1}\right)(\nabla_{y'}p - f')\cdot\nu = 0 \quad \text{on } \partial\omega. \end{cases}$$

Moreover, the distribution w is given by (3.9).

(iii) If $\lambda = 0$, then $\quad v(y) = \dfrac{(y_3^2-1)}{2\mu}(\nabla_{y'}p(y') - f'(y')), \qquad a.e.\ y \in \Omega,$

where p satisfies $\quad -\Delta_{y'}p = -\operatorname{div}_{y'}f'$ *in* $\omega, \qquad \dfrac{\partial p}{\partial \nu} = f'\cdot\nu$ *on* $\partial\omega$.

Moreover, the distribution w is zero.

Remark 3.2 *An analogous result to Theorem 3.1 is proved in [11] where it is studied the Stokes and Navier-Stokes systems with the slip condition on a rough boundary for an open set of $\mathbb{R}^3$ of fixed height. The functions $(\widehat{\phi}^i, \widehat{q}^i)$ are the same functions which appear in [11] to describe the behavior of the velocity and the pressure near the rough boundary. Moreover, it is proved there that $D_z\widehat{\phi}^i$, $\widehat{q}^i$ belong to $L^r(Z'\times(0,+\infty))^{3\times 3}$ and $L^r(Z'\times(0,+\infty))$ respectively for every $r \geq 1$ and have exponential decay at infinity.*

Remark 3.3 *For $\lambda = +\infty$, Theorem 3.1 shows that u_ε, p_ε behave as if in (3.4) we had assumed that Γ_ε was the plane boundary $\{x_3 = 0\}$ and that the boundary condition on Γ_ε was*

$$u_\varepsilon \in W^\perp \times \{0\} \quad \text{on } \Gamma_\varepsilon, \quad \partial_3 u'_\varepsilon \in W. \tag{3.10}$$

In particular, if W agrees with $\mathbb{R}^2$ (which is true except if $\Psi(z_1, z_2)$ does not depend on z_1 and/or z_2) we deduce that the slip condition in (3.4) is equivalent to the adherence condition $u_\varepsilon = 0$ on $\{x_3 = 0\}$.

For $\lambda \in (0, +\infty)$, Theorem 3.1 shows that the asymptotic behavior of u_ε and p_ε is the same that if Γ_ε was the plane boundary $\{x_3 = 0\}$ and the boundary condition on Γ_ε was

$$u_{\varepsilon,3} = 0 \quad \text{on } \Gamma_\varepsilon, \quad -\mu\partial_3 u'_\varepsilon + \lambda^2 R u'_\varepsilon = 0 \quad \text{on } \Gamma_\varepsilon, \tag{3.11}$$

i.e. although the roughness is not strong enough to deduce that the slip condition on Γ_ε is equivalent to (3.10), it is sufficient to provide the friction coefficient $\lambda^2 R u'_\varepsilon$ in (3.11).

For $\lambda = 0$, the roughness is so weak that u_ε and p_ε behave as if Γ_ε was plane.

The critical size $\lambda \in (0, +\infty)$ can be considered as the general one. In fact, the cases $\lambda = 0$ and $\lambda = +\infty$ can be obtained from this one by taking the limit when λ tends to zero and infinity respectively.

Remark 3.4 *In the cases $\lambda = 0$ or $+\infty$, we can prove that the convergences in (3.7)-(3.8) are strong. In fact, assuming ω smooth enough (for example C^2), we can show that defining $\bar{u}_\varepsilon$, $\bar{p}_\varepsilon$ by*

$$\bar{u}_\varepsilon(x) = \left(\varepsilon^2 v(x', \frac{x_3}{\varepsilon}), 0\right), \quad \bar{p}_\varepsilon(x) = p(x') \quad \text{a.e. } x \in \Omega_\varepsilon,$$

we have $\displaystyle \frac{1}{\varepsilon^4} \fint_{\Omega_\varepsilon} |u_\varepsilon - \bar{u}_\varepsilon|^2 dx \to 0, \quad \frac{1}{\varepsilon^2} \fint_{\Omega_\varepsilon} |D(u_\varepsilon - \bar{u}_\varepsilon)|^2 dx \to 0, \quad \fint_{\Omega_\varepsilon} |p_\varepsilon - \bar{p}_\varepsilon|^2 dx \to 0.$

In the critical case $\lambda \in (0, +\infty)$, the above assertion still holds replacing $\bar{u}_\varepsilon$ by

$$\bar{u}_\varepsilon(x) = \left(\varepsilon^2 v(x', \frac{x_3}{\varepsilon}), 0\right) + \lambda\varepsilon\sqrt{\varepsilon r_\varepsilon}\left(v_1(x', 0)\widehat{\phi}^1(\frac{x}{r_\varepsilon}) + v_2(x', 0)\widehat{\phi}^2(\frac{x}{r_\varepsilon})\right).$$

Remark 3.5 *The proof of Theorem 3.1 is based on the unfolding method ([3], [9], [12]). For a.e. $x' \in \mathbb{R}^2$, we define $\kappa(x') \in \mathbb{Z}^2$ by $x' \in \kappa(x') + Z'$. Then, to study the behavior of $(u_\varepsilon, p_\varepsilon)$ near Γ_ε, the idea is to study the behavior of the sequences $\widehat{u}_\varepsilon$, $\widehat{p}_\varepsilon$ defined as*

$$\widehat{u}_\varepsilon(x', z) = u_\varepsilon\left(r_\varepsilon\kappa(\frac{x'}{r_\varepsilon}) + r_\varepsilon z', r_\varepsilon z_3\right), \quad \widehat{p}_\varepsilon(x', z) = p_\varepsilon\left(r_\varepsilon\kappa(\frac{x'}{r_\varepsilon}) + r_\varepsilon z', r_\varepsilon z_3\right),$$

for a.e. $x' \in \omega$, $z' \in Z'$, $-(\delta_\varepsilon/r_\varepsilon)\Psi(z') < z_3 < 1/r_\varepsilon$. This is similar to the idea used in [11], but here it is necessary to combine this change of variables with (3.6).

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Chapter 4

Asymptotic behavior of the Navier-Stokes system in a thin domain with the slip condition on a slightly rough boundary

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Abstract.

We study the asymptotic behavior of the solutions of the Navier-Stokes system in a thin domain Ω_ε of thickness ε satisfying the slip boundary condition on a periodic rough set $\Gamma_\varepsilon \subset \partial\Omega_\varepsilon$ of period r_ε and amplitude δ_ε, with $\delta_\varepsilon \ll r_\varepsilon \ll \varepsilon$. We prove that the limit behavior as ε goes to zero depends on the limit λ of $(\delta_\varepsilon\sqrt{\varepsilon})/(r_\varepsilon\sqrt{r_\varepsilon})$. Namely, If $\lambda = +\infty$, the roughness is so strong that the fluid behaves as if we had imposed the adherence condition on Γ_ε. If $\lambda = 0$, the roughness is too weak and the fluid behaves as if Γ_ε was plane. Finally, if $\lambda \in (0, +\infty)$, the solution is strong enough to make appear a new friction term in the limit.

4.1 Introduction

The most usual boundary condition for a viscous fluid surrounded by an impermeable wall is the adherence condition, which establishes that the velocity of the fluid vanishes on the boundary. However, some other boundary conditions can be imposed which may seem more adequate from a physical point of view. The slip or Navier's boundary condition asserts that the normal component of the velocity is zero (i.e. the fluid cannot across the wall) and that the wall exerts a tangential friction force. Indeed, for a rugous boundary, it has been proved that both adherence and slip conditions are equivalent. Thus, the adherence condition is justified because of the existence of microasperities on the boundary.

The equivalence between the adherence and slip condition was proved in [9] for a rugous boundary described by the equation

$$x_3 = -\varepsilon\Psi\left(\frac{x_1}{\varepsilon}, \frac{x_2}{\varepsilon}\right) \quad \forall (x_1, x_2) \in \omega,$$

with $\varepsilon > 0$ devoted to converge to zero, ω a Lipschitz bounded open set of $\mathbb{R}^2$ and Ψ a smooth periodic function such that

$$\mathrm{Span}(\{\nabla\Psi(z') : z' \in \mathbb{R}^2\}) = \mathbb{R}^2. \tag{4.1}$$

Imposing slip conditions of this boundary, it was shown that in the limit the velocity of the fluid satisfies the adherence condition. These results were generalized in [1] to the case of a non periodic boundary described by

$$x_3 = -\Psi_\varepsilon\left(x_1, x_2\right) \quad \forall (x_1, x_2) \in \omega,$$

with Ψ_ε Lipschitz functions such that the support of the Young's measure associated to $\nabla\Psi_\varepsilon$ contains at least two independent vectors.

In [11] it was considered the case of a viscous fluid satisfying the slip condition on a slightly rugous wall described by the equation

$$x_3 = -\delta_\varepsilon\Psi\left(\frac{x_1}{\varepsilon}, \frac{x_2}{\varepsilon}\right) \quad \forall (x_1, x_2) \in \omega,$$

with $\delta_\varepsilon/\varepsilon$ converging to zero and Ψ smooth and periodic. It was proved that if $\delta_\varepsilon/\varepsilon^{\frac{3}{2}}$ tends to infinity and (4.1) holds, then the equivalence between the adherence and the slip conditions still holds, while if $\delta_\varepsilon/\varepsilon^{\frac{3}{2}}$ tends to zero then the fluid behaves as if the boundary was plane. The case $\delta_\varepsilon \sim \varepsilon^{\frac{3}{2}}$ is the critical size where the roughness is not so large to imply the adherence condition but it is enough to make appears a new friction force in the limit.

A general result about the form of the limit equation for the Navier-Stokes system satisfying the slip condition on a (non-necessarily periodic) rough boundary has been obtained in [9].

Our aim in the present paper is to extend the results in [11] to the case of a domain of small height ε. Namely, for a Lipschitz bounded open set $\omega \subset \mathbb{R}^2$ and a function Ψ in $W^{2,\infty}_{loc}(\mathbb{R}^2)$, periodic of period $Z' = (-1/2, 1/2)^2$, we will consider the open set Ω_ε given by

$$\Omega_\varepsilon = \left\{ x = (x_1, x_2, x_3) \in \omega \times \mathbb{R} :\, -\delta_\varepsilon \Psi\left(\frac{x_1}{r_\varepsilon}, \frac{x_2}{r_\varepsilon}\right) < x_3 < \varepsilon \right\},$$

where the parameters r_ε, δ_ε are positive and satisfy

$$\lim_{\varepsilon\to 0} \frac{r_\varepsilon}{\varepsilon} = 0, \quad \lim_{\varepsilon\to 0} \frac{\delta_\varepsilon}{r_\varepsilon} = 0.$$

Assuming a viscous fluid governed by the Navier-Stokes equation and satisfying the slip condition on the rough boundary

$$\Gamma_\varepsilon = \left\{ x = (x_1, x_2, x_3) \in \omega \times \mathbb{R} :\, x_3 = -\delta_\varepsilon \Psi\left(\frac{x_1}{r_\varepsilon}, \frac{x_2}{r_\varepsilon}\right) \right\},$$

we show that its pressure and velocity are asymptotically equivalent to the solutions of a Reynolds system which depend of the value of λ given by

$$\lambda = \lim_{\varepsilon\to 0} \frac{\delta_\varepsilon}{r_\varepsilon} \sqrt{\frac{\varepsilon}{r_\varepsilon}}.$$

The role of λ is similar to the one of the limit of $\delta_\varepsilon / r_\varepsilon$ in [11]. Namely, we have:

- If $\lambda = \infty$ and (4.1) holds, then the fluid behaves as if we imposed an adherence condition.
- If $\lambda \in (0, +\infty)$, then the roughness is not strong enough to give the adherence condition in the limit but it is enough to obtain a friction force term in the limit.
- If $\lambda = 0$ the roughness is so weak that the fluid behaves as if the wall was plane.

We remark that in fact, for $\varepsilon = 1$, λ is the limit of $\delta_\varepsilon / r_\varepsilon^{\frac{3}{2}}$, which is coherent with [11].

As in [11], the proof of our results is based on the unfolding method [4], [11], [16], but here it is necessary to combine it with a rescaling in the height variable, in order to work with a domain of height one.

The results obtained in the present paper were announced in [12] for the case of the Stokes system.

We finish this introduction refering to [2], [3], [5], [6], [7], [8], [17], for the study of the behavior viscous fluids in rugous domains satisfying different boundary conditions of the ones considered in the present paper.

4.2 Notation

The elements $x \in \mathbb{R}^3$ will be decomposed as $x = (x', x_3)$ with $x' \in \mathbb{R}^2$, $x_3 \in \mathbb{R}$.

By Z', we denote the unitary cube of $\mathbb{R}^2$, $Z' = (-\frac{1}{2}, \frac{1}{2})^2$, and by $\widehat{Q}$ the set $\widehat{Q} = Z' \times (0, +\infty)$. For every $M > 0$ we write $\widehat{Q}_M = Z' \times (0, M)$.

We use the index $\#$ to mean periodicity with respect Z', for example $L^2_\#(Z')$ denotes the space of functions $u \in L^2_{loc}(\mathbb{R}^2)$ which are Z'-periodic, while $L^2_\#(\widehat{Q})$ denotes the space of functions $\hat{u} \in L^2_{loc}(\mathbb{R}^2 \times (0, +\infty))$ such that

$$\int_{\widehat{Q}} |\hat{u}|^2 dz < +\infty, \quad \hat{u}(z' + k', z_3) = \hat{u}(z), \quad \forall k' \in \mathbb{Z}^2, \quad \text{a.e. } z \in \mathbb{R}^2 \times (0, +\infty).$$

For a bounded measurable set $\Theta \subset \mathbb{R}^N$, we denote by $L^2_0(\Theta)$ the space of functions of $L^2(\Theta)$ with zero mean value in Θ.

We denote by ε, r_ε and δ_ε three positive parameters devoted to tend to zero such that

$$\lim_{\varepsilon \to 0} \frac{\delta_\varepsilon}{r_\varepsilon} = 0, \quad \lim_{\varepsilon \to 0} \frac{r_\varepsilon}{\varepsilon} = 0. \tag{4.2}$$

For a connected Lipschitz open set $\omega \subset \mathbb{R}^2$ and a function $\Psi \in W^{2,\infty}_\#(Z')$, $\Psi \geq 0$ in Z', we define

$$\Omega_\varepsilon = \left\{ x \in \mathbb{R}^3 : \; x' \in \omega, \, -\delta_\varepsilon \Psi\left(\frac{x'}{r_\varepsilon}\right) < x_3 < 1 \right\}, \tag{4.3}$$

$$\Omega_\varepsilon^- = \Omega_\varepsilon \cap (\omega \times (-\infty, 0)), \quad \Omega_\varepsilon^+ = \Omega_\varepsilon \cap (\omega \times (0, +\infty)), \tag{4.4}$$

$$\Gamma_\varepsilon = \left\{ x \in \mathbb{R}^3 : \; x' \in \omega, \, x_3 = -\delta_\varepsilon \Psi\left(\frac{x'}{r_\varepsilon}\right) \right\}, \tag{4.5}$$

$$\widetilde{\Omega}_\varepsilon = \left\{ y \in \mathbb{R}^3 : \; y' \in \omega, \, -\frac{\delta_\varepsilon}{\varepsilon} \Psi\left(\frac{y'}{r_\varepsilon}\right) < y_3 < 1 \right\}, \tag{4.6}$$

$$\widetilde{\Gamma}_\varepsilon = \left\{ y \in \mathbb{R}^3 : \; y' \in \omega, \, y_3 = -\frac{\delta_\varepsilon}{\varepsilon} \Psi\left(\frac{y'}{r_\varepsilon}\right) \right\}, \tag{4.7}$$

$$\Omega = \omega \times (0, 1), \quad \Gamma = \omega \times \{0\}. \tag{4.8}$$

We denote by ν the outside unitary orthogonal vector to Ω_ε on $\partial\Omega_\varepsilon$.

Our aim here is to study the asymptotic behavior of a viscous fluid in the thin domain Ω_ε, which satisfies slip boundary condition on Γ_ε. For this purpose we will use an adaptation of the unfolding method [16]. We refer to [11], [13], [18] for other different applications of this method and its relation with the two-scale convergence method of G. Nguetsend and G. Allaire ([1], [19]). In order to apply the method, we will need the following notation.

For $k' \in \mathbb{Z}^2$ and $\rho > 0$, we denote

$$C_\rho^{k'} = \rho k' + \rho Z', \quad \Omega_\rho^{k'} = \Omega_\varepsilon \cap (C_\rho^{k'} \times (-\infty, 1)).$$

We define $\kappa : \mathbb{R}^2 \to \mathbb{Z}^2$ by

$$\kappa(x') = k' \Leftrightarrow x' \in C_1^{k'}.$$

Remark that κ is well defined up to a set of zero measure in $\mathbb{R}^2$ (the set $\cup_{k' \in \mathbb{Z}^2} \partial C_1^{k'}$). Moreover, for every $\rho > 0$, we have

$$\kappa\left(\frac{x'}{\rho}\right) = k' \Leftrightarrow x' \in C_\rho^{k'}.$$

For a.e. $x' \in \mathbb{R}^2$ we define $C_{r_\varepsilon}(x')$ as the cube $C_{r_\varepsilon}^{k'}$ such that $x' \in C_{r_\varepsilon}^{k'}$.

For every $\alpha > 0$, we take

$$\omega_\alpha = \{x \in \omega : \text{dist}(x, \partial\omega) > \alpha\} \tag{4.9}$$

and

$$I_{\alpha, r_\varepsilon} = \{k' \in \mathbb{Z}^2 : C_{r_\varepsilon}^{k'} \cap \omega_\alpha \neq \emptyset\}.$$

We denote by $\mathcal{V}$ the space of functions $\hat{v} : \mathbb{R}^2 \times (0, +\infty) \to \mathbb{R}$ such that

$$\hat{v} \in H_\#^1(\widehat{Q}_M), \quad \forall M > 0, \quad \nabla \hat{v} \in L_\#^2(\widehat{Q})^3.$$

It is a Hilbert space endowed with the norm $\|\cdot\|_\mathcal{V}$ defined by

$$\|\hat{v}\|_\mathcal{V}^2 = \|\hat{v}\|_{L^2(Z' \times \{0\}}^2 + \|\nabla \hat{v}\|_{L^2(\widehat{Q})^3}^2.$$

We denote by O_ε a generic real sequence which tends to zero with ε and can change line to line.

We denote by C a generic positive constant which can change line to line.

4.3 Main results

In this section we describe the asymptotic behavior of a viscous fluid in the geometry Ω_ε described in Section 4.2 and satisfying slip conditions on Γ_ε. The proof of the corresponding results will be given in Section 4.4.

We consider a sequence $(u_\varepsilon, p_\varepsilon) \in H^1(\Omega_\varepsilon)^3 \cap L_0^2(\Omega_\varepsilon)$, which satisfies

$$\begin{cases} -\mu\Delta u_\varepsilon + \nabla p_\varepsilon + (u_\varepsilon \cdot \nabla) u_\varepsilon = f_\varepsilon \quad \text{in } \Omega_\varepsilon, \\ \operatorname{div} u_\varepsilon = 0 \quad \text{in } \Omega_\varepsilon, \\ u_\varepsilon \nu = 0, \ \ T\left(\dfrac{\partial u_\varepsilon}{\partial \nu} + \dfrac{\gamma}{\varepsilon} u_\varepsilon\right) = 0 \qquad \text{on } \Gamma_\varepsilon, \\ u_\varepsilon = 0 \text{ on } \omega \times \{1\}, \end{cases} \tag{4.10}$$

where $\mu > 0$, $\gamma \geq 0$ are two fixed constants, and f_ε is defined by

$$f_\varepsilon(x) = \tilde{f}\left(x', \frac{x_3}{\varepsilon}\right), \quad \text{a.e. } x \in \Omega_\varepsilon, \tag{4.11}$$

for a given $\tilde{f} \in L^2(\omega \times (-1,1))^3$.

Because we are interested in showing that the behavior of the solutions of $(u_\varepsilon, p_\varepsilon)$ close to Γ_ε does not depend on the boundary conditions imposed in $\partial\Omega_\varepsilon \cap (\partial\omega \times \mathbb{R})$, we have preferred to not explicite any boundary conditions on this set. Adding some type of boundary conditions to (4.10) we have the following existence result of solution.

Theorem 4.1 *We consider $\mu > 0$, $\gamma \geq 0$ and f_ε as above. Then, adding one of the following conditions (some other conditions are also possible)*

i)

$$u_\varepsilon = 0 \quad \textit{on } \partial\Omega_\varepsilon \cap (\partial\omega \times \mathbb{R}). \tag{4.12}$$

ii)

$$u_\varepsilon \nu = 0, \quad T\left(\frac{\partial u_\varepsilon}{\partial \nu} + \kappa_\varepsilon u_\varepsilon\right) = 0 \qquad \textit{on } \partial\Omega_\varepsilon \cap (\partial\omega \times \mathbb{R}), \tag{4.13}$$

with $k_\varepsilon \geq 0$.

iii)

$$\begin{cases} \omega \textit{ is a rectangle,} \\ u_\varepsilon, \ \dfrac{\partial u_\varepsilon}{\partial \nu} - p_\varepsilon \nu \textit{ are periodic of period } \omega \textit{ with respect to } x'. \end{cases} \tag{4.14}$$

problem (4.10) has at least a solution $(u_\varepsilon, p_\varepsilon)$ in $H^1(\Omega_\varepsilon)^3 \times L^2_0(\Omega_\varepsilon)$. Moreover, there exists $C > 0$, which does not depend on ε, such that

$$\|Du_\varepsilon\|_{L^2(\Omega_\varepsilon)^{3\times 3}} + \|\nabla p_\varepsilon\|_{H^{-1}(\Omega_\varepsilon)^3} \leq C\varepsilon^{3/2}. \tag{4.15}$$

In the following, instead of assuming some boundary conditions on $\partial\Omega_\varepsilon \cap (\partial\omega \times \mathbb{R})$, we are going to focus in studying the asymptotic behavior of a solution $(u_\varepsilon, p_\varepsilon)$ of (4.10) which satisfies (4.15). For this purpose, as usual, we use a dilatation in the variable x_3 in order to have the functions defined in the open set of fixed heigh $\widetilde{\Omega}_\varepsilon$ defined by (4.6). Namely, we define $\tilde{u}_\varepsilon \in H^1(\widetilde{\Omega}_\varepsilon)^3$, $\tilde{p}_\varepsilon \in L^2_0(\widetilde{\Omega}_\varepsilon)$ as the functions obtained from u_ε and p_ε, respectively, by using the change of variables

$$y' = x', \quad y_3 = \frac{x_3}{\varepsilon}, \tag{4.16}$$

that is

$$\tilde{u}_\varepsilon(y) = u_\varepsilon(y', \varepsilon y_3), \quad \tilde{p}_\varepsilon(y) = p_\varepsilon(y', \varepsilon y_3), \qquad \text{a.e. } y \in \widetilde{\Omega}_\varepsilon. \tag{4.17}$$

The goal becomes in describing the asymptotic behavior of these new sequences $\tilde{u}_\varepsilon$, $\tilde{p}_\varepsilon$. This is given in the following theorem.

Theorem 4.2 *Let* $(u_\varepsilon, p_\varepsilon) \in H^1(\Omega_\varepsilon)^3 \times L^2_0(\Omega_\varepsilon)$ *be a solution of (4.10) such that (4.15) holds, and let* $\tilde{u}_\varepsilon, \tilde{p}_\varepsilon$ *be given by (4.17). Then, there exist* $\tilde{u}' \in H^1(0,1;L^2(\omega))^2$, $\tilde{w} \in H^2(0,1;H^{-1}(\omega))$ *and* $\tilde{p} \in H^1(\Omega)$, *which does not depend on* y_3 *and has null mean value in* Ω, *such that, up to a subsequence,*

$$\frac{\tilde{u}_\varepsilon}{\varepsilon} \rightharpoonup 0 \text{ in } H^1(\Omega)^3,\ \frac{\tilde{u}_\varepsilon}{\varepsilon^2} \rightharpoonup (\tilde{u}', 0) \text{ in } H^1(0,1;L^2(\omega))^3,\ \frac{\tilde{u}_{\varepsilon,3}}{\varepsilon^3} \rightharpoonup \tilde{w} \text{ in } H^2(0,1;H^{-1}(\omega)), \tag{4.18}$$

$$\frac{1}{\varepsilon^2} div_{y'}(\tilde{u}'_\varepsilon) + \frac{1}{\varepsilon^3}\partial_{y_3}\tilde{u}_{\varepsilon,3} \to 0 \text{ in } L^2(\Omega), \tag{4.19}$$

$$\tilde{p}_\varepsilon \rightharpoonup \tilde{p} \text{ in } L^2(\Omega), \quad \frac{\partial_{y_3}\tilde{p}_\varepsilon}{\varepsilon} \rightharpoonup \tilde{f}_3 \text{ in } H^{-1}(\Omega), \tag{4.20}$$

and they satisfy

$$\begin{cases} -\mu\partial^2_{y_3y_3}\tilde{u}' + \nabla_{y'}\tilde{p} = \tilde{f}' \quad \text{in } \Omega, \\ div_{y'}\tilde{u}' + \partial_{y_3}\tilde{w} = 0 \quad \text{in } \Omega, \\ \tilde{u}'(1) = 0, \quad \tilde{w}(0) = \tilde{w}(1) = 0. \end{cases} \tag{4.21}$$

Moreover, according to the value of the limit (it always exists at least for a subsequence)

$$\lambda = \lim_{\varepsilon\to 0} \frac{\delta_\varepsilon}{r_\varepsilon}\sqrt{\frac{\varepsilon}{r_\varepsilon}} \in [0, +\infty], \tag{4.22}$$

$\tilde{u}'$ *satisfies the following boundary condition on* Γ*:*

(i) If $\lambda = +\infty$, *then defining*

$$W = Span\left(\{\nabla\Psi(z') \,:\, z' \in Z'\}\right), \tag{4.23}$$

and $P_{W^\perp}$ *the orthogonal projection from* $\mathbb{R}^2$ *to* $W^\perp$, *we have that* $\tilde{u}'$ *satisfies*

$$\mu\partial_{y_3}\tilde{u}' + \gamma\tilde{u}' \in W, \quad \tilde{u}' \in W^\perp \quad \text{on } \Gamma. \tag{4.24}$$

(ii) If $\lambda \in (0, +\infty)$, *then defining* $(\widehat{\phi}^i, \widehat{q}^i) \in \mathcal{V}^3 \times L^2_\#(Z' \times \mathbb{R}^+)$, $i = 1, 2$, *as a solution of*

$$\begin{cases} -\mu\Delta_z\widehat{\phi}^i + \nabla_z\widehat{q}^i = 0 \text{ in } \mathbb{R}^2 \times \mathbb{R}^+, \\ div_z\widehat{\phi}^i = 0 \text{ in } \mathbb{R}^2 \times \mathbb{R}^+, \\ \widehat{\phi}^i_3 = \partial_{z_i}\Psi \text{ on } \mathbb{R}^2 \times \{0\}, \\ \partial_{z_3}(\widehat{\phi}^i)' = 0 \text{ on } \mathbb{R}^2 \times \{0\}, \end{cases} \tag{4.25}$$

and $R \in \mathbb{R}^{2\times 2}$ by

$$R_{ij} = \mu \int_{Z'\times(0,+\infty)} D_z\widehat{\phi}^i : D_z\widehat{\phi}^j\, dz, \quad \forall\, i,j \in \{1,2\} \tag{4.26}$$

we have that $\tilde{u}'$ satisfies

$$\mu\partial_{y_3}\tilde{u}' + \gamma\tilde{u}' + \lambda^2 R\tilde{u}' = 0 \quad \text{on } \Gamma. \tag{4.27}$$

(iii) If $\lambda = 0$, then we have that $\tilde{u}'$ satisfies

$$\mu\partial_{y_3}\tilde{u}' + \gamma\tilde{u}' = 0 \quad \text{on } \Gamma. \tag{4.28}$$

From (4.21), (4.24), (4.27) and (4.28), as usual in the asymptotic study of fluids in thin domains, it is easy to see that the limit pressure $\tilde{p}$ is solution of a Reynolds problem. To characterize completely the function $\tilde{p}$ (and $\tilde{u}'$, $\tilde{w}$) we need to impose additional boundary conditions to the solutions of (4.10). In this sense we have the following result, which is an immediate consequence of Theorem 4.2. For the sake of simplicity, we assume that $\tilde{f}'$ does not depend on y_3.

Corollary 4.3 *Let $(u_\varepsilon, p_\varepsilon) \in H^1(\Omega_\varepsilon)^3 \times L^2_0(\Omega_\varepsilon)$ be a solution of (4.10) satisfying one of the boundary conditions given by (4.12) or (4.13). Let us assume that $\tilde{f}'$ does not depend on y_3. Then, depending on the value of λ defined by (4.22), the functions $\tilde{u}'$, $\tilde{w}$, $\tilde{p}$ given by Theorem 4.2 satisfy:*

(i) If $\lambda = +\infty$, then $\tilde{p}$ is the solution of the Reynolds system

$$\begin{cases} -div_{y'}\left(\left(\frac{1}{3}I + (1+\frac{\gamma}{\mu})^{-1}P_{W^\perp}\right)(\nabla_{y'}\tilde{p} - \tilde{f}')\right) = 0 \quad \text{in } \omega, \\ \left(\left(\frac{1}{3}I + (1+\frac{\gamma}{\mu})^{-1}P_{W^\perp}\right)(\nabla_{y'}\tilde{p} - \tilde{f}')\right)\nu = 0 \quad \text{on } \partial\omega. \end{cases} \tag{4.29}$$

Moreover, $\tilde{u}'$ and $\tilde{w}$ are given by

$$\tilde{u}'(y) = \frac{(y_3 - 1)}{2\mu}\left(y_3 I + (1+\frac{\gamma}{\mu})^{-1}P_{W^\perp}\right)\left(\nabla_{y'}\tilde{p}(y') - \tilde{f}'(y')\right), \quad a.e.\ y \in \Omega, \tag{4.30}$$

$$\tilde{w}(y) = -\int_0^{y_3} div_{y'}\tilde{u}'(y', s)\, ds, \quad a.e.\ y \in \Omega. \tag{4.31}$$

(ii) If $\lambda \in (0,+\infty)$, then $\tilde{p}$ is the solution of the Reynolds system

$$\begin{cases} -\mathrm{div}_{y'}\left(\left(\frac{1}{3}I + \left((1+\frac{\gamma}{\mu})I + \frac{\lambda^2}{\mu}R\right)^{-1}\right)(\nabla_{y'}p - \tilde{f}')\right) = 0 \quad \text{in } \omega, \\ \left(\left(\frac{1}{3}I + \left((1+\frac{\gamma}{\mu})I + \frac{\lambda^2}{\mu}R\right)^{-1}\right)(\nabla_{y'}\tilde{p} - \tilde{f}')\right)\nu = 0 \quad \text{on } \partial\omega. \end{cases} \tag{4.32}$$

Moreover, $\tilde{u}'$ is given by

$$\tilde{u}'(y) = \frac{(y_3-1)}{2\mu}\left(y_3 I + \left((1+\frac{\gamma}{\mu})I + \frac{\lambda^2}{\mu}R\right)^{-1}\right)\left(\nabla_{y'}\tilde{p}(y') - \tilde{f}'(y')\right), \quad \text{a.e. } y \in \Omega, \tag{4.33}$$

and $\tilde{w}$ is defined by (4.31).

(iii) If $\lambda = 0$, then $\tilde{p}$ is the solution of the Reynolds system

$$\begin{cases} -\mathrm{div}_{y'}\left(\left(\frac{1}{3} + \left(1+\frac{\gamma}{\mu}\right)^{-1}\right)\left(\nabla_{y'}\tilde{p} - \tilde{f}'\right)\right) = 0 \ \text{in } \omega. \\ \left(\left(\frac{1}{3} + \left(1+\frac{\gamma}{\mu}\right)^{-1}\right)\left(\nabla_{y'}\tilde{p} - \tilde{f}'\right)\right)\nu = 0 \quad \text{on } \partial\omega. \end{cases} \tag{4.34}$$

Moreover, $\tilde{u}'$ is given by

$$\tilde{u}'(y) = \frac{1}{2\mu}\left(y_3^2 + \left(1+\frac{\gamma}{\mu}\right)^{-1}\right)(\nabla_{y'}\tilde{p}(y') - \tilde{f}'(y')), \quad \text{a.e. } y \in \Omega, \tag{4.35}$$

and $\tilde{w}$ is the null function.

Remark 4.4 *An analogous result to Theorem 4.2 is proved in [11] where the Stokes and Navier-Stokes systems are studied with slip conditions on a rough boundary for an open set of $\mathbb{R}^3$. The functions $\widehat{\phi}^i$ and $\widehat{q}^i$ are the same functions which appear in [11] to describe the behavior of the velocity and the pressure near the rough boundary. Moreover, it is proved there that $D_z\widehat{\phi}^i$, $\widehat{q}^i$ belong to $L^r(Z' \times (0,+\infty))^{3\times 3}$ and $L^r(Z' \times (0,+\infty))$ respectively for every $r \geq 2$ and have exponential decay at infinity. In particular it shows that the matrix R given by* (4.26) *is well defined.*

Remark 4.5 *For $\lambda = +\infty$, Theorem 4.2 shows that u_ε, p_ε behave as if in (4.10) we had assumed that Γ_ε was the plane boundary $\{x_3 = 0\}$ and that the boundary condition on Γ_ε was*

$$u_\varepsilon \in W^\perp \times \{0\} \quad \text{on } \Gamma_\varepsilon, \quad \mu\partial_3 u'_\varepsilon + \gamma u'_\varepsilon \in W. \tag{4.36}$$

In particular, if W is $\mathbb{R}^2$ (which is true up if $\Psi(z_1, z_2)$ does not depend on z_1 and/or z_2) we deduce that the slip condition in (4.10) is similar to the adherence condition $u_\varepsilon = 0$ on $x_3 = 0$ (or on the rough boundary Γ_ε).

For $\lambda \in (0, +\infty)$, Theorem 4.2 shows that the asymptotic behavior of u_ε and p_ε is the same that if Γ_ε was the plane boundary $\{x_3 = 0\}$ and the boundary condition on Γ_ε was

$$u_{\varepsilon,3} = 0 \quad on\ \Gamma_\varepsilon, \quad \mu\partial_3 u'_\varepsilon + \gamma u'_\varepsilon + \lambda^2 R u'_\varepsilon = 0 \quad on\ \Gamma_\varepsilon, \tag{4.37}$$

i.e. although the roughness is not strong enough to deduce that the slip condition on Γ_ε is equivalent to (4.36), it is sufficient to provide the friction coefficient $\lambda^2 R u'_\varepsilon$ in (4.37).

For $\lambda = 0$, the roughness is so weak that u_ε and p_ε behave as if Γ_ε was the plane boundary $x_3 = 0$ and the boundary condition on Γ_ε was

$$u_{\varepsilon,3} = 0 \quad on\ \Gamma_\varepsilon, \quad \mu\partial_3 u'_\varepsilon + \gamma u'_\varepsilon = 0 \quad on\ \Gamma_\varepsilon. \tag{4.38}$$

The critical size $\lambda \in (0, +\infty)$ can be considered as the general one. In fact, the cases $\lambda = 0$, $\lambda = +\infty$ can be obtained taking from this one taking the limit when λ tends to zero and infinity respectively.

Theorem 4.2 is analogous to the result proved in [11] for a fluid with fixed height. In [11] the critical size is $\delta_\varepsilon \approx r_\varepsilon^{3/2}$, which agrees with the critical size in the present paper $\delta_\varepsilon \varepsilon^{1/2} \approx r_\varepsilon^{3/2}$ when $\varepsilon = 1$. Remark that the expression (4.22) for λ does not only depend on the parameters δ_ε, r_ε which define Γ_ε but also on the height ε of Ω_ε. This is due to the fact that far of the rugous boundary the behavior of the fluid is different from the corresponding one in [11].

Remark 4.6 *If in Theorem 4.2, we also assume that one of the conditions (4.12)-(4.14) holds, then, assuming that there exists the limit λ given by (4.22), we deduce that* (4.18)-(4.20) *hold without extracting any subsequence.*

The following theorem (corrector result) provides approximations of u_ε, Du_ε and p_ε in the strong topology of $L^2(\Omega_\varepsilon)$.

Theorem 4.7 *Let $(u_\varepsilon, p_\varepsilon) \in H^1(\Omega_\varepsilon)^3 \times L^2_0(\Omega_\varepsilon)$ be a solution of* (4.10) *such that* (4.15) *holds, and let $\tilde{u}_\varepsilon$, $\tilde{p}_\varepsilon$ be defined by* (4.17)*. Let us assume that there exist $\tilde{u}' \in H^1(0,1; L^2(\omega))^2$, $\tilde{w} \in H^2(0,1; H^{-1}(\omega))$ and $\tilde{p} \in L^2_0(\Omega)$, where $\tilde{p}$ does not depend on y_3, which satisfy* (4.18)-(4.20)*. We also assume that there exists the limit λ given by* (4.22)*. Then, we have*

(i)

$$\lim_{\varepsilon\to 0}\left(\frac{1}{\varepsilon^5}\int_{\Omega_\varepsilon^-} |u_\varepsilon|^2 dx + \frac{1}{\varepsilon^5}\int_{\Omega_\varepsilon^+}\left(|u'_\varepsilon - \varepsilon^2 \tilde{u}'(x', \frac{x_3}{\varepsilon})|^2 + |u_{\varepsilon,3}|^2\right) dx\right) = 0, \tag{4.39}$$

$$\lim_{\varepsilon\to 0}\left(\frac{1}{\varepsilon}\int_{\Omega_\varepsilon^-}|p_\varepsilon|^2\eta\,dx+\frac{1}{\varepsilon}\int_{\Omega_\varepsilon^+}|p_\varepsilon-\tilde p(x')|^2\eta\,dx\right)=0, \tag{4.40}$$

for every $\eta\in C_c^\infty(\omega)$.

(ii) If $\lambda=0$ or $+\infty$, then we have

$$\begin{aligned}&\lim_{\varepsilon\to 0}\frac{1}{\varepsilon^3}\int_{\Omega_\varepsilon^-}|Du_\varepsilon|^2dx=0,\\&\lim_{\varepsilon\to 0}\frac{1}{\varepsilon^3}\int_{\Omega_\varepsilon^+}\left(|\nabla u_{\varepsilon,3}|^2+|D_{x'}u'_\varepsilon|^2+|\partial_{x_3}u'_\varepsilon(x)-\varepsilon\partial_{y_3}\tilde u'\left(x',\frac{x_3}{\varepsilon}\right)|^2\right)\eta\,dx=0,\end{aligned} \tag{4.41}$$

for every $\eta\in C_c^\infty(\omega)$.

(iii) If $\lambda\in(0+\infty)$, taking $\widehat{\phi}^i$, $i=1,2$, as a solution of (4.25) *and defining $\hat u:\omega\times(\mathbb{R}^2\times\mathbb{R}^+)\to\mathbb{R}^3$ by*

$$\hat u(x',z)=-\lambda\tilde u_1(x',0)\widehat{\phi}^1(z)-\lambda\tilde u_2(x',0)\widehat{\phi}^2(z), \tag{4.42}$$

for a.e. $(x',z)\in\omega\times(\mathbb{R}^2\times\mathbb{R}^+)$, then we have

$$\begin{aligned}&\lim_{\varepsilon\to 0}\frac{1}{\varepsilon^3}\int_{\Omega_\varepsilon^-}|Du_\varepsilon|^2dx=0,\\&\lim_{\varepsilon\to 0}\frac{1}{\varepsilon^3}\int_{\Omega_\varepsilon^+}\left|\nabla u_{\varepsilon,3}-\frac{\varepsilon\sqrt{\varepsilon}}{\sqrt{r_\varepsilon}}\fint_{C_{r_\varepsilon}(x')}\nabla_z\hat u_3\left(s',\frac{x}{r_\varepsilon}\right)ds'\right|^2\eta\,dx=0,\\&\lim_{\varepsilon\to 0}\frac{1}{\varepsilon^3}\int_{\Omega_\varepsilon^+}\left|D_{x'}u'_\varepsilon-\frac{\varepsilon\sqrt{\varepsilon}}{\sqrt{r_\varepsilon}}\fint_{C_{r_\varepsilon}(x')}D_{z'}\hat u'\left(s',\frac{x}{r_\varepsilon}\right)ds'\right|^2\eta\,dx=0,\\&\lim_{\varepsilon\to 0}\frac{1}{\varepsilon^3}\int_{\Omega_\varepsilon^+}\left|\partial_{x_3}u'_\varepsilon-\varepsilon\partial_{y_3}\tilde u'\left(x',\frac{x_3}{\varepsilon}\right)-\frac{\varepsilon\sqrt{\varepsilon}}{\sqrt{r_\varepsilon}}\fint_{C_{r_\varepsilon}(x')}\partial_{z_3}\hat u'\left(s',\frac{x}{r_\varepsilon}\right)ds'\right|^2\eta\,dx=0,\end{aligned} \tag{4.43}$$

for every $\eta\in C_c^\infty(\omega)$.

Remark 4.8 *If in Theorem 4.7, we also assume that $(u_\varepsilon,p_\varepsilon)$ satisfies one of the assumptions (4.12)-(4.14), then in (4.40), (4.41) and* (4.43) *we can take $\eta=1$.*

Remark 4.9 *The last terms in (4.43) for the corrector of Du_ε in Ω_ε^+ is a kind of corrector very usual when we apply the unfolding method (see e.g. [11], [13], [16]). If we assume additional smoothness properties for $\tilde u'$, for instance that $\tilde u'$ belongs to $H^1(\Omega)^2$ (this holds if*

$\tilde{f}' \in H^1(\Omega)^2$, *see Corollary 4.3), then we can rewrite the last three equalities in (4.43) as*

$$\lim_{\varepsilon\to 0}\frac{1}{\varepsilon^3}\int_{\Omega_\varepsilon^+}\left|\nabla u_{\varepsilon,3}-\frac{\varepsilon\sqrt{\varepsilon}}{\sqrt{r_\varepsilon}}\nabla_z\hat{u}_3\left(x',\frac{x}{r_\varepsilon}\right)\right|^2\eta\,dx=0,$$
$$\lim_{\varepsilon\to 0}\frac{1}{\varepsilon^3}\int_{\Omega_\varepsilon^+}\left|D_{x'}u'_\varepsilon-\frac{\varepsilon\sqrt{\varepsilon}}{\sqrt{r_\varepsilon}}D_{z'}\hat{u}'\left(x',\frac{x}{r_\varepsilon}\right)\right|^2\eta\,dx=0,$$
$$\lim_{\varepsilon\to 0}\frac{1}{\varepsilon^3}\int_{\Omega_\varepsilon^+}\left|\partial_{x_3}u'_\varepsilon-\varepsilon\partial_{y_3}\tilde{u}'\left(x',\frac{x_3}{\varepsilon}\right)-\frac{\varepsilon\sqrt{\varepsilon}}{\sqrt{r_\varepsilon}}\partial_{z_3}\hat{u}'\left(x',\frac{x}{r_\varepsilon}\right)\right|^2\eta\,dx=0,$$

for every $\eta \in C_c^\infty(\omega)$.

4.4 Proofs of the results

Proof of Theorem 4.1. It is a consequence of Proposition 4.10 below. □

Proposition 4.10 *There exists a constant* $c > 0$ *independent of* ε *such that if* ε *is small enough, then:*

(i) If $\zeta_\varepsilon \in H^{-1}(\Omega_\varepsilon)^3$ *satisfies*

$$<\zeta_\varepsilon, w_\varepsilon>=0, \quad \forall w_\varepsilon \in H_0^1(\Omega_\varepsilon)^3 \text{ with } div(w_\varepsilon)=0 \text{ in } \Omega_\varepsilon, \tag{4.44}$$

then there exist $p_\varepsilon^1 \in H^1(\Omega_\varepsilon)$, *which does not depend on the variable* x_3 *and has null mean value in* Ω_ε, *and* $p_\varepsilon^0 \in L_0^2(\Omega_\varepsilon)$ *such that* $\zeta_\varepsilon = \nabla p_\varepsilon^1 + \nabla p_\varepsilon^0$ *in* Ω_ε *and*

$$\varepsilon^{\frac{3}{2}}\|p_\varepsilon^1\|_{H^1(\Omega_\varepsilon)}+\|p_\varepsilon^0\|_{L^2(\Omega_\varepsilon)}\le c\|\zeta_\varepsilon\|_{H^{-1}(\Omega_\varepsilon)^3}. \tag{4.45}$$

(ii) For every $p_\varepsilon \in L_0^2(\Omega_\varepsilon)$ *there exists* $v_\varepsilon \in H_0^1(\Omega_\varepsilon)^3$ *satisfying* $div\, v_\varepsilon = p_\varepsilon$ *in* Ω_ε *and*

$$\|v_\varepsilon\|_{H_0^1(\Omega_\varepsilon)^3}\le\frac{c}{\varepsilon}\|p_\varepsilon\|_{L^2(\Omega_\varepsilon)}. \tag{4.46}$$

(iii) For every $w_\varepsilon \in H^1(\Omega_\varepsilon)$ *with* $w_\varepsilon = 0$ *on* $\omega \times \{\varepsilon\}$, *we have*

$$\|w_\varepsilon\|_{L^6(\Omega_\varepsilon)}\le c\|\nabla w_\varepsilon\|_{L^2(\Omega_\varepsilon)^3}. \tag{4.47}$$

To prove Proposition 4.10 we need some previous lemmas.

Lemma 4.11 *Let us denote by* Q_ε, *with* $\varepsilon > 0$, *the set*

$$Q_\varepsilon=\left\{x\in\mathbb{R}^3 : x'\in\varepsilon Y',\, -\delta_\varepsilon\Psi(\frac{x'}{r_\varepsilon})<x_3<\varepsilon\right\}. \tag{4.48}$$

There exists a positive constant c *independent of* ε *such that if* ε *is small enough, then:*

(i) For every $p_\varepsilon \in L^2_0(Q_\varepsilon)$ there exists $v_\varepsilon \in H^1_0(Q_\varepsilon)^3$ satisfying

$$div\, v_\varepsilon = p_\varepsilon \text{ in } Q_\varepsilon \tag{4.49}$$

$$\|v_\varepsilon\|_{H^1_0(Q_\varepsilon)^3} \le c\|p_\varepsilon\|_{L^2(Q_\varepsilon)}. \tag{4.50}$$

(ii) For every $p_\varepsilon \in L^2(Q_\varepsilon)$ and every $S_\varepsilon \subset Q_\varepsilon$ measurable, we have

$$\left\| p_\varepsilon - \fint_{S_\varepsilon} p_\varepsilon\, dx \right\|_{L^2(Q_\varepsilon)} \le c\left(1 + \sqrt{\frac{|Q_\varepsilon|}{|S_\varepsilon|}}\right) \|\nabla p_\varepsilon\|_{H^{-1}(Q_\varepsilon)^3}. \tag{4.51}$$

(iii) For every $w_\varepsilon \in H^1(Q_\varepsilon)$ with $w_\varepsilon = 0$ on $\varepsilon Y' \times \{\varepsilon\}$, we have

$$\|w_\varepsilon\|_{L^6(Q_\varepsilon)} \le c\|\nabla w_\varepsilon\|_{L^2(Q_\varepsilon)^3}. \tag{4.52}$$

Proof. Along the proof we use the application $\eta_\varepsilon : Q_\varepsilon \to \mathbb{R}^3$ defined by $\eta_\varepsilon(x) = (x', x_3 + h_\varepsilon(x))$ with

$$h_\varepsilon(x) = \frac{\delta_\varepsilon \Psi(\frac{x'}{r_\varepsilon})(\varepsilon - x_3)}{\varepsilon + \delta_\varepsilon \Psi(\frac{x'}{r_\varepsilon})},$$

which transforms Q_ε in the cube $Y_\varepsilon = (0,\varepsilon)^3$. It is easy to check that h_ε satisfies

$$|h_\varepsilon| \le C\delta_\varepsilon,\ |\nabla_x h_\varepsilon| \le C\frac{\delta_\varepsilon}{r_\varepsilon} \quad \text{in } Q_\varepsilon, \quad \forall \varepsilon > 0. \tag{4.53}$$

Given p_ε and using the change of variables $y = \eta_\varepsilon(x)$, the equation

$$\operatorname{div} v_\varepsilon = p_\varepsilon \quad \text{in } Q_\varepsilon,$$

is equivalent to

$$\operatorname{div}_y \check{v}_\varepsilon = \check{p}_\varepsilon - H_\varepsilon(\check{v}_\varepsilon) \quad \text{in } Y_\varepsilon, \tag{4.54}$$

where we have denoted $\check{v}_\varepsilon(y) = v_\varepsilon(\eta_\varepsilon^{-1}(y))$, $\check{p}_\varepsilon(y) = p_\varepsilon(\eta_\varepsilon^{-1}(y))$ and

$$H_\varepsilon(\check{v}_\varepsilon)(y) = \partial_{y_3}\check{v}'_\varepsilon \nabla_{x'} h_\varepsilon(\eta_\varepsilon^{-1}(y)) + \partial_{y_3}\check{v}_{\varepsilon,3}\partial_{x_3} h_\varepsilon(\eta_\varepsilon^{-1}(y)), \quad \text{a.e. } y \in Y_\varepsilon.$$

Since Y_ε is homothetic to the unit cube Y, it is well known that there exists a linear continuous operator $L_\varepsilon : L^2_0(Y_\varepsilon) \to H^1_0(Y_\varepsilon)^3$ such that

$$\operatorname{div} L_\varepsilon(\check{\pi}_\varepsilon) = \check{\pi}_\varepsilon \quad \text{in } Y_\varepsilon, \quad \|L_\varepsilon(\check{\pi}_\varepsilon)\|_{H^1_0(Y_\varepsilon)^3} \le C\|\check{\pi}_\varepsilon\|_{L^2(Y_\varepsilon)}, \quad \forall\, \check{\pi}_\varepsilon \in L^2_0(Y_\varepsilon), \tag{4.55}$$

with $C > 0$ independent of ε. Thanks to (4.2), (4.53) and the Banach fixed point theorem, we have that for ε small enough there exists a unique $\check{\varphi}_\varepsilon \in H^1_0(Y_\varepsilon)^3$ satisfying the equation

$$\check{\varphi}_\varepsilon = L_\varepsilon\Big(\check{p}_\varepsilon - \fint_{Y_\varepsilon} \check{p}_\varepsilon dy - H_\varepsilon(\check{\varphi}_\varepsilon) + \fint_{Y_\varepsilon} H_\varepsilon(\check{\varphi}_\varepsilon) dy\Big),$$

or equivalently

$$\operatorname{div}_y \check{\varphi}_\varepsilon = \check{p}_\varepsilon - H_\varepsilon(\check{\varphi}_\varepsilon) - \fint_{Y_\varepsilon} (\check{p}_\varepsilon - H_\varepsilon(\check{\varphi}_\varepsilon))\, dy \quad \text{in } Y_\varepsilon. \tag{4.56}$$

Moreover, by (4.53) and (4.55), $\check{\varphi}_\varepsilon$ satisfies

$$\|\check{\varphi}_\varepsilon\|_{H_0^1(Y_\varepsilon)^3} \leq C\left(\|\check{p}_\varepsilon\|_{L^2(Y_\varepsilon)} + \frac{\delta_\varepsilon}{r_\varepsilon}\|\check{\varphi}_\varepsilon\|_{H_0^1(Y_\varepsilon)^3}\right),$$

i.e.

$$\|\check{\varphi}_\varepsilon\|_{H_0^1(Y_\varepsilon)^3} \leq C\|\check{p}_\varepsilon\|_{L^2(Y_\varepsilon)}, \quad \forall \varepsilon > 0 \text{ small enough.} \tag{4.57}$$

Let us check that $v_\varepsilon \in H_0^1(Q_\varepsilon)^3$ defined by $v_\varepsilon(x) = \check{\varphi}_\varepsilon(\eta_\varepsilon(x))$ satisfies the result. Using the change of variables $y = \eta_\varepsilon(x)$ in (4.56), we obtain that v_ε satisfies

$$\operatorname{div} v_\varepsilon = p_\varepsilon - \fint_{Y_\varepsilon} (\check{p}_\varepsilon - H_\varepsilon(\check{\varphi}_\varepsilon))\, dy \quad \text{in } Q_\varepsilon.$$

Integrating this equality and using that $p_\varepsilon \in L_0^2(\Omega_\varepsilon)$, we deduce that the mean value in Y_ε of $\check{p}_\varepsilon - H_\varepsilon(\check{\varphi}_\varepsilon)$ is zero, and therefore that v_ε satisfies (4.49). From (4.57), taking into account that $\|\eta_\varepsilon\|_{W^{1,\infty}(Q_\varepsilon)^3} \leq C$ and $\|\eta_\varepsilon^{-1}\|_{W^{1,\infty}(Y_\varepsilon)^3} \leq C$ for some C which does not depend on ε, we deduce that v_ε also satisfies (4.50).

Inequality (4.51) in case $S_\varepsilon = Q_\varepsilon$ is an immediate consequence of property (i). The general case follows easily from the previous one and the estimate

$$\left|\fint_{Q_\varepsilon} p_\varepsilon\, ds - \fint_{S_\varepsilon} p_\varepsilon\, ds\right|^2 \leq \frac{1}{|S_\varepsilon|}\int_{Q_\varepsilon} |p_\varepsilon - \fint_{Q_\varepsilon} p_\varepsilon\, ds|^2 dx.$$

The proof of (iii) follows using again the change of variables $y = \eta_\varepsilon(x)$ and that $H^1(Y_\varepsilon)$ is continuously imbedded in $L^6(Y_\varepsilon)$, with $\|\check{w}_\varepsilon\|_{L^6(Y_\varepsilon)} \leq C\|\nabla\check{w}_\varepsilon\|_{L^2(Y_\varepsilon)^3}$, for every $\check{w}_\varepsilon \in H^1(Y_\varepsilon)$ with $\check{w}_\varepsilon = 0$ on $\{y_3 = 0\}$, for some positive constant C independent of ε (because Y_ε is homothetic to Y). □

Lemma 4.12 *There exists $C > 0$ such that, if $\varepsilon > 0$ is small enough, then for every p_ε in $L^2(\Omega_\varepsilon)$ there exist $p_\varepsilon^1 \in H^1(\Omega_\varepsilon)$, which does not depend on the variable x_3, and $p_\varepsilon^0 \in L^2(\Omega_\varepsilon)$ such that*

$$p_\varepsilon = p_\varepsilon^1 + p_\varepsilon^0 \quad \text{in } \Omega_\varepsilon, \tag{4.58}$$

$$\varepsilon^{\frac{3}{2}}\|\nabla p_\varepsilon^1\|_{L^2(\omega)^2} + \|p_\varepsilon^0\|_{L^2(\Omega_\varepsilon)} \leq C\|\nabla p_\varepsilon\|_{H^{-1}(\Omega_\varepsilon)^3}. \tag{4.59}$$

Proof. Firstly, we assume that ω is the rectangle of sides parallel to the coordinate axes given by $\omega = (0, a) \times (0, b)$, for some $a, b > 0$, and we prove that there exist $p_\varepsilon^1 \in H^1(\Omega_\varepsilon \cap (\omega_\varepsilon \times \mathbb{R}))$

(see (4.9) for the definition of ω_α, with $\alpha > 0$), which does not depend on the variable x_3, and $p_\varepsilon^0 \in L^2(\Omega_\varepsilon \cap (\omega_\varepsilon \times \mathbb{R}))$ such that

$$p_\varepsilon = p_\varepsilon^1 + p_\varepsilon^0 \quad \text{in } \Omega_\varepsilon \cap (\omega_\varepsilon \times \mathbb{R}), \tag{4.60}$$

$$\varepsilon^{\frac{3}{2}} \|\nabla p_\varepsilon^1\|_{L^2(\omega_\varepsilon)^2} + \|p_\varepsilon^0\|_{L^2(\Omega_\varepsilon \cap (\omega_\varepsilon \times \mathbb{R}))} \leq C \|\nabla p_\varepsilon\|_{H^{-1}(\Omega_\varepsilon)^3}. \tag{4.61}$$

We take p_ε in $L^2(\Omega_\varepsilon)$ and we denote by w_ε the solution of the Dirichlet problem

$$-\Delta w_\varepsilon = \nabla p_\varepsilon \ \text{ in } \Omega_\varepsilon, \quad w_\varepsilon \in H_0^1(\Omega_\varepsilon),$$

which satisfies

$$\|w_\varepsilon\|_{H_0^1(\Omega_\varepsilon)} = \|\nabla p_\varepsilon\|_{H^{-1}(\Omega_\varepsilon)^3}. \tag{4.62}$$

For every $\varepsilon > 0$, we define $m_1^\varepsilon, m_2^\varepsilon \in \mathbb{N}$ by

$$m_1^\varepsilon = \max\{m_1 \in \mathbb{N} : \frac{\varepsilon}{2} m_1 \in [0, a - \frac{\varepsilon}{2}]\}, \quad m_2^\varepsilon = \max\{m_2 \in \mathbb{N} : \frac{\varepsilon}{2} m_2 \in [0, b - \frac{\varepsilon}{2}]\},$$

and we denote by G^ε the open subset of ω given by

$$G^\varepsilon = (\frac{\varepsilon}{2}, \frac{\varepsilon}{2} m_1^\varepsilon) \times (\frac{\varepsilon}{2}, \frac{\varepsilon}{2} m_2^\varepsilon).$$

Observe that $\omega_\varepsilon \subset G^\varepsilon \subset \omega_{\varepsilon/2}$, for every $\varepsilon > 0$. We denote by $\mathcal{T}^\varepsilon = \{T_i^\varepsilon\}$ the triangulation of G^ε which consists of the triangles whose vertices are either the points $\frac{\varepsilon}{2} m'$, $\frac{\varepsilon}{2} m' + \frac{\varepsilon}{2}(e_1 + e_2)$, $\frac{\varepsilon}{2} m' + \frac{\varepsilon}{2} e_1$, or the points $\frac{\varepsilon}{2} m'$, $\frac{\varepsilon}{2} m' + \frac{\varepsilon}{2}(e_1 + e_2)$, $\frac{\varepsilon}{2} m' + \frac{\varepsilon}{2} e_2$, for some $m' = (m_1, m_2) \in \mathbb{N}^2$ with $1 \leq m_i < m_i^\varepsilon$, $i = 1, 2$. Then we define p_ε^1 as the unique element in $H^1(G^\varepsilon)$ which is affine in every triangle T_i^ε of $\mathcal{T}^\varepsilon$ and satisfies

$$p_\varepsilon^1(\frac{\varepsilon}{2} m') = \fint_{\Omega_{\frac{\varepsilon}{2}}^{m'}} p_\varepsilon(s)\, ds, \quad \forall m' \in \mathbb{N}^2 \text{ such that } \frac{\varepsilon}{2} m' \in \overline{G^\varepsilon}.$$

We fix a triangle T_i^ε of $\mathcal{T}^\varepsilon$. We assume that the vertices of T_i^ε are the points $\frac{\varepsilon}{2} m'$, $\frac{\varepsilon}{2} m' + \frac{\varepsilon}{2}(e_1 + e_2)$, $\frac{\varepsilon}{2} m' + \frac{\varepsilon}{2} e_1$ for some $m' = (m_1, m_2) \in \mathbb{N}^2$ with $1 \leq m_i < m_i^\varepsilon$, $i = 1, 2$ (the case where the vertices are the points $\frac{\varepsilon}{2} m'$, $\frac{\varepsilon}{2} m' + \frac{\varepsilon}{2}(e_1 + e_2)$, $\frac{\varepsilon}{2} m' + \frac{\varepsilon}{2} e_2$ is completely analogous). Integrating in $\Omega_\varepsilon^{m'}$ the inequality

$$|p_\varepsilon^1(\frac{\varepsilon}{2} m') - p_\varepsilon^1(\frac{\varepsilon}{2} m' + \frac{\varepsilon}{2} e_1)|^2 \leq 2|p_\varepsilon^1(\frac{\varepsilon}{2} m') - p_\varepsilon(x)|^2 + 2|p_\varepsilon(x) - p_\varepsilon^1(\frac{\varepsilon}{2} m' + \frac{\varepsilon}{2} e_1)|^2, \text{ a.e. } x \in \Omega_\varepsilon^{m'},$$

applying (4.51) twice with $\Omega_\varepsilon^{m'}$ instead of Q_ε (observe that the set $\Omega_\varepsilon^{m'}$ is obtained from Q_ε by a displacement) and $S_\varepsilon = \Omega_{\frac{\varepsilon}{2}}^{m'}$ and $S_\varepsilon = \Omega_{\frac{\varepsilon}{2}}^{m'+e_1}$, we obtain

$$\varepsilon^3 |p_\varepsilon^1(\frac{\varepsilon}{2} m') - p_\varepsilon^1(\frac{\varepsilon}{2} m' + \frac{\varepsilon}{2} e_1)|^2 \leq C \|\nabla p_\varepsilon\|^2_{H^{-1}(\Omega_\varepsilon^{m'})^3} \leq C \int_{\Omega_\varepsilon^{m'}} |\nabla w_\varepsilon|^2 ds.$$

Since p_ε^1 is affine in the triangle T_i^ε, this inequality proves

$$\left|\frac{\partial p_\varepsilon^1}{\partial x_1}(x')\right|^2 \leq \frac{C}{\varepsilon^5}\int_{\Omega_\varepsilon^{m'}} |\nabla w_\varepsilon|^2 ds, \quad \text{a.e. } x' \in T_i^\varepsilon,$$

which gives

$$\int_{T_i^\varepsilon} \left|\frac{\partial p_\varepsilon^1}{\partial x_1}\right|^2 dx' \leq \frac{C}{\varepsilon^3}\int_{\Omega_\varepsilon^{m'}} |\nabla w_\varepsilon|^2 ds.$$

The same reasoning but estimating now the difference between $p_\varepsilon^1(\frac{\varepsilon}{2}m' + \frac{\varepsilon}{2}e_1)$ and $p_\varepsilon^1(\frac{\varepsilon}{2}m' + \frac{\varepsilon}{2}(e_1 + e_2))$ provides the same bound for the integral of $|\partial p_\varepsilon^1/\partial x_2|^2$ in T_i^ε. Summing these estimates for every triangle T_i^ε of $\mathfrak{T}^\varepsilon$ and using (4.62), we deduce

$$\int_{G^\varepsilon} \left|\nabla_{x'} p_\varepsilon^1\right|^2 dx' \leq \frac{C}{\varepsilon^3}\|\nabla p_\varepsilon\|^2_{H^{-1}(\Omega_\varepsilon)^3}. \tag{4.63}$$

To complete the demonstration of (4.60) and (4.61), it is enough to prove that $p_\varepsilon^0 = p_\varepsilon - p_\varepsilon^1$ satisfies

$$\|p_\varepsilon^0\|_{L^2(\Omega_\varepsilon)} \leq C\|\nabla p_\varepsilon\|_{H^{-1}(\Omega_\varepsilon)^3}. \tag{4.64}$$

Again we fix a triangle T_i^ε of $\mathfrak{T}^\varepsilon$ and we assume that the its vertices are the points $V_1 = \frac{\varepsilon}{2}m'$, $V_2 = \frac{\varepsilon}{2}m' + \frac{\varepsilon}{2}(e_1 + e_2)$, $V_3 = \frac{\varepsilon}{2}m' + \frac{\varepsilon}{2}e_1$. As p_ε^1 is affine in T_i^ε, it is easy to see that

$$p_\varepsilon^1(x') = \sum_{j=1}^{3} \lambda_j(x') p_\varepsilon^1(V_j), \ \text{a.e. } x' \in T_i^\varepsilon,$$

for three no-negative functions λ_1, λ_2, λ_3 which satisfy

$$\sum_{j=1}^{3} \lambda_j(x) = 1, \ \text{a.e. } x \in T_i^\varepsilon.$$

Then, thanks again to (4.51), we have

$$\int_{\Omega_\varepsilon \cap (T_i^\varepsilon \times \mathbb{R})} |p_\varepsilon(x) - p_\varepsilon^1(x')|^2 dx \leq \sum_{j=1}^{3} \int_{\Omega_\varepsilon \cap (T_i^\varepsilon \times \mathbb{R})} |p_\varepsilon(x) - p_\varepsilon^1(V_j)|^2 dx$$

$$\leq \sum_{j=1}^{3} \int_{\Omega_\varepsilon^{m'}} |p_\varepsilon(x) - p_\varepsilon^1(V_j)|^2 dx \leq 3\|\nabla p_\varepsilon\|^2_{H^{-1}(\Omega_\varepsilon^{m'})^3} \leq 3\int_{\Omega_\varepsilon^{m'}} |\nabla w_\varepsilon|^2 ds.$$

From (4.62), summing last estimate for every T_i^ε we get (4.64).

Next we continue with the case $\omega = (0,a)\times(0,b)$, for some $a,b>0$, but now we show that (4.60) and (4.61) still hold true if we replace ω_ε by

$$\omega_\varepsilon^0 = \{x\in\omega : \operatorname{dist}(x,\partial\omega\setminus\{x_2=0\})<\varepsilon\}.$$

Unlike ω_ε, whose points are at a positive distance from $\partial\omega$, we have $\omega_\varepsilon^0\cap\partial\omega\neq\emptyset$. To prove the inequalities for ω_ε^0, we consider the set

$$\check{G}^\varepsilon = (\frac{\varepsilon}{2},\frac{\varepsilon}{2}m_1^\varepsilon)\times(0,\frac{\varepsilon}{2}m_2^\varepsilon),$$

and its triangulation $\check{\mathfrak{T}}^\varepsilon = \{T_i^\varepsilon\}$ consisting of the triangles whose vertices are either the points $\frac{\varepsilon}{2}m'$, $\frac{\varepsilon}{2}m'+\frac{\varepsilon}{2}(e_1+e_2)$, $\frac{\varepsilon}{2}m'+\frac{\varepsilon}{2}e_1$, or the points $\frac{\varepsilon}{2}m'$, $\frac{\varepsilon}{2}m'+\frac{\varepsilon}{2}(e_1+e_2)$, $\frac{\varepsilon}{2}m'+\frac{\varepsilon}{2}e_2$, for some $m'=(m_1,m_2)\in\mathbb{N}^2$ with $1\le m_1<m_1^\varepsilon$, $0\le m_2<m_2^\varepsilon$. Then we extend p_ε^1 given above to the set $\check{G}^\varepsilon$ in a such way that it is affine in every triangle of $\check{\mathfrak{T}}^\varepsilon$ and in the vertices on $\partial\omega$ it takes the values

$$p_\varepsilon^1(\frac{\varepsilon}{2}m_1,0) = \fint_{\Omega_{\frac{\varepsilon}{2}}^{m'}\cap\Omega_\varepsilon} p_\varepsilon(s)\,ds,\quad \forall m_1\in\{1,\dots,m_1^\varepsilon\}.$$

Now the decomposition and the estimates with $\check{G}^\varepsilon$ are obtained following exactly the same reasoning employed above with G^ε.

Finally, in order to prove the result in the case of an arbitrary smooth open set ω, it is enough to consider a system of local charts and apply the inequalities obtained when ω is a rectangle. □

Proof of Proposition 4.10.

Given $\zeta_\varepsilon\in H^{-1}(\Omega_\varepsilon)^3$ satisfying (4.44), it is well known the existence of $p_\varepsilon\in L_0^2(\Omega_\varepsilon)$ such that $\zeta_\varepsilon=\nabla p_\varepsilon$ in Ω_ε. Then (i) is an immediate consequence of Lemma 4.12 applied to this p_ε. Remark that

$$\|p_\varepsilon^1+p_\varepsilon^0\|_{L^2(\Omega_\varepsilon)}\le\frac{c}{\varepsilon}\|\nabla(p_\varepsilon^1+p_\varepsilon^0)\|_{H^{-1}(\Omega_\varepsilon)^3}.$$

Statement (ii) follows easily from (i) and last estimate. Finally, to prove (iii) it is enough to consider a recovering of Ω_ε by subsets of the form (4.48) and apply (4.52). □

Theorem 4.1 gives the existence of at least a solution of problem (4.10) which satisfies (4.15). Applying Lemma 4.12 we obtain the following estimate for p_ε which will be very useful to obtain its limit behavior.

Corollary 4.13 *Let $(u_\varepsilon,p_\varepsilon)\in H^1(\Omega_\varepsilon)^3\times L_0^2(\Omega_\varepsilon)$ a solution of (4.10). If ε is small enough, then there exist $p_\varepsilon^1\in H^1(\Omega_\varepsilon)$, which does not depend on the variable x_3 and has null mean value in Ω_ε, and $p_\varepsilon^0\in L_0^2(\Omega_\varepsilon)$ such that $\nabla p_\varepsilon=\nabla p_\varepsilon^1+\nabla p_\varepsilon^0$ in $H^{-1}(\Omega_\varepsilon)^3$ and*

$$\|p_\varepsilon^1\|_{H^1(\Omega_\varepsilon)}\le C\sqrt{\varepsilon},\qquad \|p_\varepsilon^0\|_{L^2(\Omega_\varepsilon)}\le C\varepsilon^{\frac{3}{2}}, \tag{4.65}$$

with C a positive constant independent of p_ε and ε.

The following two lemmas are compactness results for sequences $u_\varepsilon \in H^1(\Omega_\varepsilon)^3$, $p_\varepsilon \in L^2_0(\Omega_\varepsilon)$, not necessarily solutions of (4.10), when we write them in the new variables (4.16).

Lemma 4.14 *Let u_ε be in $H^1(\Omega_\varepsilon)^3$ such that $u_\varepsilon = 0$ on $\omega \times \{\varepsilon\}$, $u_\varepsilon \nu = 0$ on Γ_ε, and there exists a constant C independent of ε satisfying*

$$\fint_{\Omega_\varepsilon} |Du_\varepsilon|^2 dx \leq C\varepsilon^2, \tag{4.66}$$

$$\fint_{\Omega_\varepsilon} |div(u_\varepsilon)|^2 dx \leq C\varepsilon^4. \tag{4.67}$$

Let us define $\tilde{u}_\varepsilon \in H^1(\tilde{\Omega}_\varepsilon)^3$ by (4.17). Then, there exist $\tilde{u}' \in H^1(0,1;L^2(\omega))^2$, $\tilde{w} \in H^2(0,1;H^{-1}(\omega))$ and $\tilde{\pi} \in L^2(\Omega)$ such that

$$\begin{gathered} \tilde{u}'(1) = 0 \ \ in \ L^2(\omega), \quad \tilde{w}(0) = \tilde{w}(1) = 0 \ \ in \ H^{-1}(\omega), \\ div_{y'}(\tilde{u}') + \partial_{y_3}\tilde{w} = \tilde{\pi} \ \ in \ H^1(0,1;H^{-1}(\omega)), \end{gathered} \tag{4.68}$$

and, up to a subsequence,

$$\frac{\tilde{u}_\varepsilon}{\varepsilon} \rightharpoonup 0 \ \ in \ H^1(\Omega)^3, \ \frac{\tilde{u}_\varepsilon}{\varepsilon^2} \rightharpoonup (\tilde{u}', 0) \ \ in \ H^1(0,1;L^2(\omega))^3, \ \frac{\tilde{u}_{\varepsilon,3}}{\varepsilon^3} \rightharpoonup \tilde{w} \ \ in \ H^2(0,1;H^{-1}(\omega)), \tag{4.69}$$

$$\frac{1}{\varepsilon^2} div_{y'}(\tilde{u}'_\varepsilon) + \frac{1}{\varepsilon^3}\partial_{y_3}\tilde{u}_{\varepsilon,3} \rightharpoonup \tilde{\pi} \ \ in \ L^2(\Omega). \tag{4.70}$$

Moreover, if $div(u_\varepsilon) = 0$ in Ω_ε, then $\tilde{\pi} = 0$.

Proof. Since u_ε vanishes on $\omega \times \{\varepsilon\}$, then Poincaré's inequality and (4.66) imply u_ε also satisfies

$$\frac{1}{\varepsilon}\int_{\Omega_\varepsilon} |u_\varepsilon|^2 dx \leq C\varepsilon^4.$$

Applying the change of variables (4.16) in this inequality, in (4.66) and in (4.67), we deduce that $\tilde{u}_\varepsilon$ defined by (4.17) satisfies

$$\int_{\tilde{\Omega}_\varepsilon} |\tilde{u}_\varepsilon|^2 \leq C\varepsilon^4, \quad \int_{\tilde{\Omega}_\varepsilon} \left(|\nabla_{y'}\tilde{u}_\varepsilon|^2 + \frac{1}{\varepsilon^2}|\partial_{y_3}\tilde{u}_\varepsilon|^2 \right) dx \leq C\varepsilon^2, \tag{4.71}$$

$$\int_{\tilde{\Omega}_\varepsilon} \left| \mathrm{div}_{y'}(\tilde{u}'_\varepsilon) + \frac{1}{\varepsilon}\partial_{y_3}\tilde{u}_{\varepsilon,3} \right|^2 dy \leq C\varepsilon^4. \tag{4.72}$$

Therefore we deduce there exists $\tilde{u} \in H^1(0,1;L^2(\omega))^3$, with $\tilde{u}(1) = 0$ in $L^2(\omega)$, $\tilde{\pi} \in L^2(\Omega)$ such that, up to a subsequence,

$$\frac{\tilde{u}_\varepsilon}{\varepsilon} \rightharpoonup 0 \ \ \text{in } H^1(\Omega)^3, \tag{4.73}$$

$$\frac{\tilde{u}_\varepsilon}{\varepsilon^2} \rightharpoonup \tilde{u} \ \text{ in } H^1(0,1;L^2(\omega))^3, \tag{4.74}$$

$$\frac{1}{\varepsilon^2}\mathrm{div}_{y'}(\tilde{u}'_\varepsilon) + \frac{1}{\varepsilon^3}\partial_{y_3}\tilde{u}_{\varepsilon,3} \rightharpoonup \tilde{\pi} \text{ in } L^2(\Omega). \tag{4.75}$$

Dividing (4.72) by ε^4, taking into account (4.74) and (4.75), we deduce that $\partial_{y_3}\tilde{u}_{\varepsilon,3}/\varepsilon^3$ is bounded in $H^1(0,1;H^{-1}(\omega))$. Since $\tilde{u}_{\varepsilon,3} = 0$ on $\omega \times \{1\}$, this implies that $\tilde{u}_{\varepsilon,3}/\varepsilon^3$ is bounded in $H^2(0,1;H^{-1}(\omega))$ and therefore gives the existence of $\tilde{w} \in H^2(0,1;H^{-1}(\omega))$, with $\tilde{w}(1) = 0$ in $H^{-1}(\omega)$, such that, up to a subsequence,

$$\frac{\tilde{u}_{\varepsilon,3}}{\varepsilon^3} \rightharpoonup \tilde{w} \ \text{ in } H^2(0,1;H^{-1}(\omega)), \tag{4.76}$$

and

$$\mathrm{div}_{y'}(\tilde{u}') + \partial_{y_3}\tilde{w} = \tilde{\pi} \ \text{ in } \Omega. \tag{4.77}$$

On the other hand, if we now divide (4.72) by ε^2 and pass to the limit using (4.73) and (4.74), we get that $\partial_{y_3}\tilde{u}_3 = 0$ in Ω. As $\tilde{u}_3 = 0$ on $\omega \times \{1\}$, this implies that $\tilde{u}_3 = 0$ in Ω.

Now, we consider $\eta \in C_c^\infty(\omega)$ and $\vartheta \in C^\infty(\mathbb{R})$, with $\vartheta(y_3) = \vartheta(0)$, for every $y_3 < 0$. Integrating by parts, thanks to the boundary conditions satisfied by u_ε (and therefore by $\tilde{u}_\varepsilon$), we get

$$\int_{\tilde{\Omega}_\varepsilon} \left(\frac{1}{\varepsilon^2}\mathrm{div}_{y'}(\tilde{u}'_\varepsilon) + \frac{1}{\varepsilon^3}\partial_{y_3}\tilde{u}_{\varepsilon,3}\right)\eta(y')\vartheta(y_3)dy$$
$$= -\int_{\tilde{\Omega}_\varepsilon} \frac{\tilde{u}'_\varepsilon(y)}{\varepsilon^2}\cdot\nabla_{y'}\eta(y')\vartheta(y_3)dy - \int_\Omega \frac{\tilde{u}_{\varepsilon,3}(y)}{\varepsilon^3}\eta(y')\partial_{y_3}\vartheta(y_3)dy.$$

Using that by (4.71) and (4.72) we have

$$\int_{\tilde{\Omega}_\varepsilon\setminus\Omega} |\frac{\tilde{u}'_\varepsilon}{\varepsilon^2}|dy \to 0, \quad \int_{\tilde{\Omega}_\varepsilon\setminus\Omega} |\frac{1}{\varepsilon^2}\mathrm{div}_{y'}(\tilde{u}'_\varepsilon) + \frac{1}{\varepsilon^3}\partial_{y_3}\tilde{u}_{\varepsilon,3}|dy \to 0,$$

we can write the previous equality as

$$\int_\Omega \left(\frac{1}{\varepsilon^2}\mathrm{div}_{y'}(\tilde{u}'_\varepsilon) + \frac{1}{\varepsilon^3}\partial_{y_3}\tilde{u}_{\varepsilon,3}\right)\eta(y')\vartheta(y_3)dy$$
$$= -\int_\Omega \frac{\tilde{u}'_\varepsilon(y)}{\varepsilon^2}\nabla_{y'}\eta(y')\vartheta(y_3)dy - \int_0^1 \langle\frac{\tilde{u}_{\varepsilon,3}}{\varepsilon^3},\eta\rangle_{H^{-1}(\omega),H_0^1(\omega)}\partial_{y_3}\vartheta(y_3)dy_3 + O_\varepsilon,$$

where O_ε tends to zero. Passing to the limit in this equality by means of (4.74) and (4.76), we get

$$\int_0^1 \langle\mathrm{div}_{y'}\,\tilde{u}',\eta\rangle_{H^{-1}(\omega),H_0^1(\omega)}\vartheta(y_3)dy_3 + \int_0^1 \langle\partial_{y_3}\tilde{w},\eta\rangle_{H^{-1}(\omega),H_0^1(\omega)}\vartheta(y_3)dy_3$$
$$= -\int_\Omega \tilde{u}'\nabla_{y'}\eta(y')\vartheta(y_3)dy - \int_0^1 \langle\tilde{w},\eta\rangle_{H^{-1}(\omega),H_0^1(\omega)}\partial_{y_3}\vartheta(y_3)dy_3,$$

which, by the arbitrariness of η and ϑ, proves $\tilde{w}(0) = 0$ in $H^{-1}(\omega)$.

Finally, if $\operatorname{div}(u_\varepsilon) = 0$ in Ω_ε, then the sequence in the left hand side of (4.75) is equal to zero, and thus $\tilde{\pi} = 0$ in Ω. □

Lemma 4.15 *Let p^1_ε, p^0_ε be in $H^1(\Omega_\varepsilon)$, $L^2_0(\Omega_\varepsilon)$ respectively, with p^1_ε independent of the variable x_3 and with null mean value in Ω_ε. Let us also assume that there exists a constant $C > 0$ such that*

$$\|p^1_\varepsilon\|_{H^1(\Omega_\varepsilon)} \leq C\sqrt{\varepsilon}, \qquad \|p^0_\varepsilon\|_{L^2_0(\Omega_\varepsilon)} \leq C\varepsilon^{\frac{3}{2}}, \tag{4.78}$$

for every $\varepsilon > 0$, and let us define $\tilde{p}^1_\varepsilon \in H^1(\tilde{\Omega}_\varepsilon)$, $\tilde{p}^0_\varepsilon \in L^2_0(\tilde{\Omega}_\varepsilon)$ by

$$\tilde{p}^1_\varepsilon(y') = p^1_\varepsilon(y'), \quad \tilde{p}^0_\varepsilon(y) = p^0_\varepsilon(y', \varepsilon y_3), \quad \text{a.e. } y \in \tilde{\Omega}_\varepsilon. \tag{4.79}$$

Then there exist $\tilde{p}^1 \in H^1(\Omega)$, which does not depend on y_3 and has null mean value in Ω, and $\tilde{p}^0 \in L^2_0(\Omega)$ such that, up to a subsequence,

$$\tilde{p}^1_\varepsilon \rightharpoonup \tilde{p}^1 \text{ in } H^1(\Omega), \qquad \frac{\tilde{p}^0_\varepsilon}{\varepsilon} \rightharpoonup \tilde{p}^0 \text{ in } L^2(\Omega). \tag{4.80}$$

Proof. Observe that $\tilde{p}^1_\varepsilon$ and $\tilde{p}^0_\varepsilon$ are obtained from p^1_ε and p^0_ε by using the change of variables (4.16). Applying this change in (4.78), we deduce that $\tilde{p}^1_\varepsilon$ and $\tilde{p}^0_\varepsilon/\varepsilon$ are bounded in $H^1(\Omega)$ and $L^2(\Omega)$ respectively, and then the result is immediate. □

The change of variables (4.16) does not provide the information we need about the behavior of u_ε in the part of Ω_ε close to Γ_ε. To prove Theorem 4.2, we introduce an adaptation of the unfolding method (see e.g. [4], [11], [13], [16]), which is strongly related to the two-scale convergence method, [1], [19]. For this purpose, given $u_\varepsilon \in H^1(\Omega_\varepsilon)^3$ and $\rho > 0$, we define $\hat{u}_\varepsilon$ by

$$\hat{u}_\varepsilon(x', z) = u_\varepsilon\left(r_\varepsilon \kappa\left(\frac{x'}{r_\varepsilon}\right) + r_\varepsilon z', r_\varepsilon z_3\right), \tag{4.81}$$

for a.e. $(x', z) \in \omega_\rho \times \hat{Z}_\varepsilon$, with

$$\hat{Z}_\varepsilon = \left\{z' \in Z' \times \mathbb{R} : -\frac{\delta_\varepsilon}{r_\varepsilon}\Psi(z') < z_3 < \frac{\varepsilon}{r_\varepsilon}\right\}.$$

Remark 4.16 *For $k' \in \mathbb{Z}^2$ the restriction of $\hat{u}_\varepsilon$ to $C^{k'}_{r_\varepsilon} \times \hat{Z}_\varepsilon$ does not depend on x', whereas as function of z it is obtained from u_ε by using the change of variables*

$$z' = \frac{x' - r_\varepsilon k'}{r_\varepsilon}, \quad z_3 = \frac{x_3}{r_\varepsilon}, \tag{4.82}$$

which transforms $\Omega^{k'}_{r_\varepsilon}$ into $\hat{Z}_\varepsilon$.

We need the following result whose proof is equivalent to the one of Lemma 4.1 in [11].

Lemma 4.17 *Let $v_\varepsilon \in L^2(\omega)$ be a sequence which converges weakly to a function v in $L^2(\omega)$. For $\rho > 0$, we define $\bar{v}_\varepsilon \in L^2(\omega_\rho)$ by*

$$\bar{v}_\varepsilon(x') = \frac{1}{r_\varepsilon^2} \int_{C_{r_\varepsilon}(x')} v_\varepsilon(\eta')d\eta', \quad a.e.\ x' \in \omega_\rho.$$

Then we have:

(i) For every $\tau' \in \mathbb{R}^2$, the sequence w_ε defined by

$$w_\varepsilon(x') = \sqrt{\frac{\varepsilon}{r_\varepsilon}}(\bar{v}_\varepsilon(x' + r_\varepsilon \tau') - \bar{v}_\varepsilon(x'))$$

converges to zero in the sense of distributions in ω_ρ.

(ii) If the convergence of v_ε is strong, then $\bar{v}_\varepsilon$ converges strongly to v in $L^2(\omega_\rho)$.

Lemma 4.18 *We consider a sequence $u_\varepsilon \in H^1(\Omega_\varepsilon)^3$ satisfying (4.66) and $u_\varepsilon \nu = 0$ on Γ_ε. We define $\tilde{u}_\varepsilon$ by (4.17), and we assume that there exists $\tilde{u}' \in H^1(0,1;L^2(\omega))^2$ such that $\tilde{u}'_\varepsilon/\varepsilon^2$ converges weakly to $\tilde{u}'$ in $H^1(0,1;L^2(\omega))^2$ (by Lemma 4.14, this always holds for a subsequence). We also assume that there exists the limit λ given by (4.22) and that λ belongs to $(0,+\infty]$. Then we have*

(i) If $\lambda = +\infty$, then

$$\tilde{u}'(x',0)\nabla\Psi(z') = 0, \quad a.e.\ (x',z') \in \omega \times Z'. \tag{4.83}$$

(ii) If $\lambda \in (0,+\infty)$, then there exists $\hat{u} \in L^2(\Omega;\mathcal{V}^3)$ with

$$\hat{u}_3(x',z',0) = -\lambda\nabla\Psi(z')\tilde{u}'(x',0), \quad a.e.\ (x',z') \in \omega \times Z', \tag{4.84}$$

such that for every $\rho, M > 0$, the sequence $\hat{u}_\varepsilon$ defined by (4.81) satisfies

$$\frac{1}{\varepsilon\sqrt{\varepsilon r_\varepsilon}} D_z \hat{u}_\varepsilon \rightharpoonup D_z \hat{u} \quad in\ L^2(\omega_\rho \times \widehat{Q}_M)^{3\times 3}. \tag{4.85}$$

Besides, if $div\, u_\varepsilon = 0$ in Ω, then

$$div_z \hat{u} = 0 \quad in\ \omega \times \widehat{Q}. \tag{4.86}$$

Proof. We proceed in several steps.

Step 1. Let us obtain some estimates for the sequence $\hat{u}_\varepsilon$ given by (4.81).

For $\rho, M > 0$, the definition (4.81) of $\hat{u}_\varepsilon$ and (4.66) prove for every $\varepsilon > 0$ small enough

$$\begin{aligned}\int_{\omega_\rho\times\widehat{Q}_M} |D_z\hat{u}_\varepsilon|^2 dx'dz &\le r_\varepsilon^4 \sum_{k'\in I_{\rho,\varepsilon}} \int_{\widehat{Q}_M} |Du_\varepsilon(r_\varepsilon(k'+z'), r_\varepsilon z_3)|^2 dz \\ &\le \sum_{k'\in I_{\rho,\varepsilon}} r_\varepsilon \int_{\Omega_\varepsilon^{k'}} |Du_\varepsilon|^2 dx \le r_\varepsilon \int_{\Omega_\varepsilon} |Du_\varepsilon|^2\, dx \le C r_\varepsilon \varepsilon^3.\end{aligned} \tag{4.87}$$

On the other hand, defining

$$\bar{u}_\varepsilon(x') = \frac{1}{r_\varepsilon^2}\int_{C_{r_\varepsilon}(x')} u_\varepsilon(\tau', 0)d\tau = \int_{Z'} \hat{u}_\varepsilon(x', z', 0)dz', \tag{4.88}$$

using the inequality

$$\int_{\widehat{Q}_M} |\hat{u}_\varepsilon(x', z) - \bar{u}_\varepsilon(x')|^2 dz \le C_M \int_{\widehat{Q}_M} |D_z\hat{u}_\varepsilon|^2 dz, \quad \text{a.e. } x' \in \omega_\rho, \tag{4.89}$$

where C_M does not depend on ε, and taking into account (4.87), we deduce that

$$\hat{U}_\varepsilon = \frac{\hat{u}_\varepsilon(x', z) - \bar{u}_\varepsilon}{\varepsilon\sqrt{\varepsilon r_\varepsilon}} \quad \text{is bounded in } L^2(\omega_\rho; H^1(\widehat{Q}_M)^3), \quad \forall \rho, M > 0. \tag{4.90}$$

Thus, there exists $\hat{u} : \omega \times \widehat{Q} \to \mathbb{R}^3$, such that, up to a subsequence,

$$\hat{U}_\varepsilon \rightharpoonup \hat{u} \text{ in } L^2(\omega_\rho; H^1(\widehat{Q}_M)^3), \quad \forall \rho, M > 0, \tag{4.91}$$

and then

$$\frac{1}{\varepsilon\sqrt{\varepsilon r_\varepsilon}} D_z\hat{u}_\varepsilon \rightharpoonup D_z\hat{u} \text{ in } L^2(\omega_\rho \times \widehat{Q}_M)^{3\times3}, \quad \forall \rho, M > 0. \tag{4.92}$$

By semicontinuity, these convergences and inequalities (4.87) and (4.89) (this latest one after integration in ω_ρ) give

$$\int_{\omega_\rho\times\widehat{Q}_M} |D_z\hat{u}|^2 dx'dz \le C, \quad \int_{\omega_\rho\times\widehat{Q}_M} |\hat{u}|^2 dx'dz \le C_M,$$

and by the arbitrariness of ρ and M, once we prove the Z'-periodicity of $\hat{u}$ in z' (Step 2), then

$$\hat{u} \in L^2(\omega; \mathcal{V}^3).$$

Moreover, if we also assume that $\operatorname{div} u_\varepsilon = 0$ in Ω_ε, then by definition (4.81) of $\hat{u}_\varepsilon$, we have $\operatorname{div}_z \hat{u}_\varepsilon = 0$ in $\omega_\rho \times \widehat{Q}_M$, which together with (4.92) proves

$$\operatorname{div}_z \hat{u} = 0 \quad \text{in } \omega \times \widehat{Q}. \tag{4.93}$$

Step 2. Let us prove that $\hat{u}$ is Z'-periodic in z'.

We observe that by definition (4.81) of $\hat{u}_\varepsilon$, for every $\rho, M > 0$, we have

$$\hat{u}_\varepsilon(x_1 + r_\varepsilon, x_2, -\frac{1}{2}, z_2, z_3) = \hat{u}_\varepsilon(x_1, x_2, \frac{1}{2}, z_2, z_3), \text{ a.e. } (x', z_2, z_3) \in \omega_\rho \times \left(-\frac{1}{2}, \frac{1}{2}\right) \times (0, M),$$

which implies

$$\hat{U}_\varepsilon(x_1 + r_\varepsilon, x_2, -\frac{1}{2}, z_2, z_3) - \hat{U}_\varepsilon(x_1, x_2, \frac{1}{2}, z_2, z_3) = -\frac{\bar{u}_\varepsilon(x_1 + r_\varepsilon, x_2) - \bar{u}_\varepsilon(x')}{\varepsilon\sqrt{\varepsilon r_\varepsilon}}$$
$$= -\sqrt{\frac{\varepsilon}{r_\varepsilon}} \frac{1}{r_\varepsilon^2} \int_{C_{r_\varepsilon}(x')} \frac{\tilde{u}_\varepsilon(y_1 + r_\varepsilon, y_2, 0) - \tilde{u}_\varepsilon(y', 0)}{\varepsilon^2} \, dy'.$$

Since $\tilde{u}_\varepsilon/\varepsilon^2$ is bounded in $L^2(\Gamma)^3$, we can apply Lemma 4.17-(i) to deduce that the right-hand side of the above equality tends to zero in the distribution sense in ω_ρ. Therefore, passing to the limit in the previous equation by (4.91), and taking into account the arbitrariness of ρ and M we get

$$\hat{u}(x', -\frac{1}{2}, z_2, z_3) - \hat{u}(x', \frac{1}{2}, z_2, z_3) = 0 \quad \text{a.e. } (x', z_2, z_3) \in \omega_\rho \times \left(\frac{-1}{2}, \frac{1}{2}\right) \times \mathbb{R}.$$

Analogously we can prove

$$\hat{u}(x', z_1, -\frac{1}{2}, z_3) - \hat{u}(x', z_1, \frac{1}{2}, z_3) = 0 \quad \text{a.e. } (x', z_1, z_3) \in \omega_\rho \times \left(\frac{-1}{2}, \frac{1}{2}\right) \times \mathbb{R}.$$

These equalities prove the periodicity of $\hat{u}$.

Step 3. Using the compact embedding of $H^1(\Omega)$ into $L^2(\Gamma)$ and Lemma 4.17 (ii), we have that $\bar{u}_\varepsilon/\varepsilon^2$ converges strongly to $(\tilde{u}'(x', 0), 0)$ in $L^2(\omega_\rho)^3$, for every $\rho > 0$. Thus, by (4.2) and (4.90), we deduce

$$\frac{\hat{u}_\varepsilon(x', z)}{\varepsilon^2} \to (\tilde{u}'(x', 0), 0) \quad \text{in } L^2(\omega_\rho; H^1(\widehat{Q}_M)^3), \quad \forall \rho, M > 0. \tag{4.94}$$

Step 4. For $\rho > 0$, using the change of variables (4.82), which defines $\hat{u}_\varepsilon$, in the equality $u_\varepsilon \nu = 0$ on Γ_ε, we get

$$-\frac{\delta_\varepsilon}{r_\varepsilon} \nabla_{z'} \Psi(z') \hat{u}'_\varepsilon\left(x', z', -\frac{\delta_\varepsilon}{r_\varepsilon}\Psi(z')\right) - \hat{u}_{\varepsilon,3}\left(x', z', -\frac{\delta_\varepsilon}{r_\varepsilon}\Psi(z')\right) = 0, \text{ a.e. in } \omega_\rho \times Z'. \tag{4.95}$$

Thanks to (4.87) and (4.95), we then have

$$\begin{aligned}
&\int_{\omega_\rho\times Z'}\left|\frac{\delta_\varepsilon}{r_\varepsilon}\nabla_{z'}\Psi(z')\hat{u}'_\varepsilon(x',z',0)+\hat{u}_{\varepsilon,3}(x',z',0)\right|^2 dz'dx'\\
&\quad\le \int_{\omega_\rho\times Z'}\int_{\frac{-\delta_\varepsilon}{r_\varepsilon}\Psi(z')}^{0}\frac{\delta_\varepsilon}{r_\varepsilon}\left|\frac{\delta_\varepsilon}{r_\varepsilon}\nabla_{z'}\Psi(z')\partial_{z_3}\hat{u}'_\varepsilon(x',z',t)+\partial_{z_3}\hat{u}_{\varepsilon,3}(x',z',t)\right|^2 dtdz'dx'\\
&\quad\le C\frac{\delta_\varepsilon}{r_\varepsilon}\int_\omega\int_{\hat{Q}_\varepsilon}|\partial_{z_3}\hat{u}_\varepsilon|^2dzdx'\le C\varepsilon^3\delta_\varepsilon,
\end{aligned}$$

which implies

$$\begin{aligned}
\int_{\omega_\rho\times Z'}\Bigg|&\frac{\delta_\varepsilon}{r_\varepsilon}\nabla_{z'}\Psi(z')\cdot\hat{u}'_\varepsilon(x',z',0)+\hat{u}_{\varepsilon,3}(x',z',0)\\
&-\int_{Z'}\left(\frac{\delta_\varepsilon}{r_\varepsilon}\nabla_{z'}\Psi(\tau')\hat{u}'_\varepsilon(x',\tau',0)+\hat{u}_{\varepsilon,3}(x',\tau',0)\right)d\tau'\Bigg|^2 dx'dz'\le C\varepsilon^3\delta_\varepsilon.
\end{aligned}$$

Dividing by $\varepsilon^3 r_\varepsilon$, and taking into account that $\nabla_{z'}\Psi$ has mean value zero in Z' and (4.2), we get

$$\left\{\begin{aligned}
&\int_{\omega_\rho\times Z'}\Bigg|\frac{\delta_\varepsilon}{r_\varepsilon}\sqrt{\frac{\varepsilon}{r_\varepsilon}}\nabla_{z'}\Psi(z')\frac{\hat{u}'_\varepsilon(x',z',0)}{\varepsilon^2}-\frac{\delta_\varepsilon}{r_\varepsilon}\int_{Z'}\nabla_{z'}\Psi(\tau')\left(\frac{\hat{u}'_\varepsilon(x',\tau',0)-\bar{u}'_\varepsilon(x')}{\varepsilon\sqrt{\varepsilon r_\varepsilon}}\right)d\tau'\\
&\qquad+\frac{\hat{u}_{\varepsilon,3}(x',z',0)-\bar{u}_{\varepsilon,3}(x')}{\varepsilon\sqrt{\varepsilon r_\varepsilon}}\Bigg|^2 dx'dz'\le C\frac{\delta_\varepsilon}{r_\varepsilon}\to 0,\quad \forall\rho>0.
\end{aligned}\right. \tag{4.96}$$

Depending on the values of λ, we deduce:

If $\lambda=+\infty$, statement (4.96) shows that $\frac{\delta_\varepsilon}{r_\varepsilon}\sqrt{\frac{\varepsilon}{r_\varepsilon}}\nabla_{z'}\Psi(z')\frac{\hat{u}'_\varepsilon(x',z',0)}{\varepsilon^2}$ is bounded in $L^2(\omega_\rho\times Z')$, for every $\rho>0$ and then that $\nabla_{z'}\Psi(z')\hat{u}'_\varepsilon(x',z',0)$ tends to zero in $L^2(\omega_\rho\times Z')$, for every $\rho>0$. By (4.94), this proves assertion (i) in the proof of Lemma 4.18.

If $\lambda\in(0,+\infty)$, we can pass to the limit in (4.96) to deduce (4.84). □

Lemma 4.19 *Let p^0_ε be in $L^2_0(\Omega_\varepsilon)$ satisfying*

$$\|p^0_\varepsilon\|_{L^2(\Omega_\varepsilon)}\le C\varepsilon^{\frac{3}{2}}, \tag{4.97}$$

and let us define $\hat{p}^0_\varepsilon$ by

$$\hat{p}^0_\varepsilon(x',z)=p^0_\varepsilon\left(r_\varepsilon\kappa\left(\frac{x'}{r_\varepsilon}\right)+r_\varepsilon z',r_\varepsilon z_3\right),\quad \textit{for a.e. } (x',z)\in\omega_\rho\times\hat{Z}_\varepsilon. \tag{4.98}$$

Then there exists $\hat{p}^0 \in L^2(\omega \times \widehat{Q})$ *such that, up to a subsequence,*

$$\frac{\sqrt{r_\varepsilon}}{\varepsilon\sqrt{\varepsilon}}\hat{p}^0_\varepsilon \rightharpoonup \hat{p}^0 \text{ in } L^2(\omega_\rho \times \widehat{Q}_M), \quad \forall\, M, \rho > 0. \tag{4.99}$$

Proof. For every $\rho, M > 0$, the definition of $\hat{p}^0_\varepsilon$ proves

$$\begin{aligned}\int_{\omega_\rho\times\widehat{Q}_M}|\sqrt{r}_\varepsilon\hat{p}^0_\varepsilon|^2\,dx'\,dz &\le \sum_{k'\in I_{\rho,\varepsilon}} r_\varepsilon^3\int_{\widehat{Q}_M}|p^0_\varepsilon(r_\varepsilon(k'+z'), r_\varepsilon z_3)|^2\,dz\\ &\le \sum_{k'\in I_{\rho,\varepsilon}}\int_{\Omega^{k'}_{r_\varepsilon}}|p^0_\varepsilon(x)|^2\,dx \le \int_{\Omega_\varepsilon}|p^0_\varepsilon|^2\,dx \le C\varepsilon^3,\end{aligned} \tag{4.100}$$

and then there exists $\hat{p}^0 : \omega \times \widehat{Q} \longrightarrow \mathbb{R}$ such that (4.99) holds. From (4.100), by semicontinuity, we deduce

$$\int_{\omega_\rho\times\widehat{Q}_M}|\hat{p}^0|^2 dx'\,dz \le C,$$

and for the arbitrariness of $\rho, M > 0$, this shows that $\hat{p}^0$ belongs to $L^2(\omega\times\widehat{Q})$. □

Proof of Theorem 4.2. Thanks to (4.15) and $\mathrm{div}(u_\varepsilon) = 0$ in Ω_ε, Lemma 4.14 assures the existence of $\tilde{u}' \in H^1(0,1;L^2(\omega))^2$ and $\tilde{w} \in H^2(0,1;H^{-1}(\omega))$ satisfying (4.68), (4.69) and (4.70), with $\tilde{\pi} = 0$. By Corollary 4.13, there exist $p^1_\varepsilon \in H^1(\Omega_\varepsilon)$, which does not depend on the variable x_3 and has null mean value in Ω_ε, and $p^0_\varepsilon \in L^2_0(\Omega_\varepsilon)$ such that $\nabla p_\varepsilon = \nabla p^1_\varepsilon + \nabla p^0_\varepsilon$ in $H^{-1}(\Omega_\varepsilon)^3$ and (4.65) holds. Then, applying Lemma 4.15 to these sequences, we have the existence of $\tilde{p}^1 \in H^1(\Omega)$, which does not depend on y_3 and has null mean value in Ω, and $\tilde{p}^0 \in L^2(\Omega)$ such that (4.80) holds for $\tilde{p}^1_\varepsilon$, $\tilde{p}^0_\varepsilon$ defined by (4.79). Observe that, defining $\tilde{p} = \tilde{p}^1$, then we have

$$\tilde{p}_\varepsilon \rightharpoonup \tilde{p} \quad \text{in } L^2(\Omega), \qquad \frac{1}{\varepsilon}\partial_{y_3}\tilde{p}_\varepsilon = \frac{1}{\varepsilon}\partial_{y_3}\tilde{p}^0_\varepsilon \rightharpoonup \partial_{y_3}\tilde{p}^0 \quad \text{in } H^{-1}(\Omega). \tag{4.101}$$

On the other hand, we remark that $(u_\varepsilon, p_\varepsilon)$ satisfies the variational equation

$$\left\{\begin{aligned}&\mu\int_{\Omega_\varepsilon}Du_\varepsilon : D\varphi_\varepsilon\,dx + \int_{\Omega_\varepsilon}\nabla p^1_\varepsilon\varphi_\varepsilon\,dx - \int_{\Omega_\varepsilon}p^0_\varepsilon\mathrm{div}\varphi_\varepsilon\,dx + \int_{\Omega_\varepsilon}(u_\varepsilon\cdot\nabla)u_\varepsilon\varphi_\varepsilon\,dx +\\ &\frac{\gamma}{\varepsilon}\int_{\Gamma_\varepsilon}u_\varepsilon\varphi_\varepsilon\,dx = \int_{\Omega_\varepsilon}f_\varepsilon\varphi_\varepsilon\,dx,\ \forall\varphi_\varepsilon \in H^1(\Omega_\varepsilon)^3,\ \varphi_\varepsilon\nu = 0 \text{ on } \Gamma_\varepsilon,\ \varphi_\varepsilon = 0 \text{ on } \partial\Omega_\varepsilon\setminus\Gamma_\varepsilon.\end{aligned}\right. \tag{4.102}$$

The proof of Theorem 4.2 will be carried out using suitable test functions φ_ε in (4.102).

Step 1. We take $\tilde{\varphi}_3 \in C^1_c(\omega\times(-1,1))$ with $supp(\tilde{\varphi}_3) \subset \Omega$ and define $\varphi_\varepsilon \in H^1(\Omega_\varepsilon)^3$ by

$$\varphi'_\varepsilon(x) = 0, \quad \varphi_{\varepsilon,3} = \frac{1}{\varepsilon}\tilde{\varphi}_3(x', \frac{x_3}{\varepsilon}) \qquad \forall x \in \Omega_\varepsilon.$$

It is immediate to check that we can take φ_ε as test function in (4.102) which gives

$$\begin{aligned}&\frac{\mu}{\varepsilon}\int_{\Omega_\varepsilon}\nabla_{x'}u_{\varepsilon,3}(x)\nabla_{y'}\tilde{\varphi}_3(x',\frac{x_3}{\varepsilon})\,dx+\frac{\mu}{\varepsilon^2}\int_{\Omega_\varepsilon}\partial_{x_3}u_{\varepsilon,3}(x)\partial_{y_3}\tilde{\varphi}_3(x',\frac{x_3}{\varepsilon})\,dx\\&-\frac{1}{\varepsilon^2}\int_{\Omega_\varepsilon}p^0_\varepsilon(x)\partial_{y_3}\tilde{\varphi}_3(x',\frac{x_3}{\varepsilon})\,dx+\frac{1}{\varepsilon}\int_{\Omega_\varepsilon}((u_\varepsilon\cdot\nabla)u_{\varepsilon,3})(x)\tilde{\varphi}_{\varepsilon,3}(x',\frac{x_3}{\varepsilon})\,dx\\&=\frac{1}{\varepsilon}\int_{\Omega_\varepsilon}f_{\varepsilon,3}(x)\tilde{\varphi}_3(x',\frac{x_3}{\varepsilon})\,dx.\end{aligned}$$

Using the change of variables (4.16) in this equality, taking into account that thanks to Hölder's inequality, (4.15), (4.47) and $\|\tilde{\varphi}_{\varepsilon,3}\|_{L^\infty(\Omega)}\le C$, we have

$$\left|\int_{\Omega_\varepsilon}(u_\varepsilon\cdot\nabla)u_{\varepsilon,3}\tilde{\varphi}_{\varepsilon,3}\,dx\right|\le\|u_\varepsilon\|_{L^4(\Omega_\varepsilon)}\|Du_\varepsilon\|_{L^2(\Omega_\varepsilon)^{3\times3}}\|\varphi_{\varepsilon,3}\|_{L^4(\Omega_\varepsilon)}\le\varepsilon^{10/3}. \tag{4.103}$$

then we obtain that $\tilde{u}_\varepsilon$ and $\tilde{p}^0_\varepsilon$ defined by (4.17) and (4.79) respectively satisfy

$$\begin{aligned}&\mu\int_{\Omega}\nabla_{y'}\tilde{u}_{\varepsilon,3}(y)\nabla_{y'}\tilde{\varphi}_3(y)\,dy+\frac{\mu}{\varepsilon^2}\int_{\Omega_\varepsilon}\partial_{y_3}\tilde{u}_{\varepsilon,3}(y)\partial_{y_3}\tilde{\varphi}_3(y)\,dy-\int_{\Omega_\varepsilon}\frac{\tilde{p}^0_\varepsilon}{\varepsilon}(y)\partial_{y_3}\tilde{\varphi}_3(y)\,dy\\&=\int_{\Omega_\varepsilon}\tilde{f}_3(y)\tilde{\varphi}_3(y)\,dy+O_\varepsilon.\end{aligned}$$

Passing to the limit in this inequality, thanks to (4.69) and (4.80) we deduce

$$\partial_{y_3}\tilde{p}^0=\tilde{f}_3\quad\text{ in }\Omega.$$

This and (4.101) give (4.20).

Step 2. Case $\lambda\in(0,+\infty)$.

This is the most difficult case and it will be developed in more detail.

We consider $\tilde{\varphi}'\in C^1_c(\omega\times(-1,1))^2$, $\hat{\varphi}\in C^1_c(\omega;C^1_\sharp(\hat{Q})^3)$ with $D_z\hat{\varphi}(x',z)=0$ a.e. in $\{z_3>M\}$ for some constant $M>0$, such that

$$\begin{gathered}\tilde{\varphi}'(y',y_3)=\tilde{\varphi}'(y',0)\ \text{ if } y_3\le 0,\qquad \hat{\varphi}(x',z',z_3)=\hat{\varphi}(x',z',0)\ \text{ if } z_3\le 0,\\ \lambda\nabla\Psi(z')\tilde{\varphi}'(y',0)+\hat{\varphi}_3(x',z',0)=0.\end{gathered} \tag{4.104}$$

Besides, we take $\zeta\in C^\infty(\mathbb{R})$ satisfying

$$\zeta(s)=1 \text{ if } s<\frac{1}{3},\qquad \zeta(s)=0\text{ if } s>\frac{2}{3}, \tag{4.105}$$

and a sequence of positive numbers T_ε such that

$$\lim_{\varepsilon\to 0}\frac{T_\varepsilon}{\varepsilon}=0,\qquad \lim_{\varepsilon\to 0}\frac{r_\varepsilon^4}{\varepsilon T_\varepsilon}=0. \tag{4.106}$$

Then, we define $\varphi_\varepsilon\in H^1(\Omega_\varepsilon)^3$ by

$$\begin{cases}\varphi'_\varepsilon(x)=\dfrac{1}{\varepsilon}\tilde{\varphi}'\left(x',\dfrac{x_3}{\varepsilon}\right)+\dfrac{\delta_\varepsilon}{\lambda\varepsilon r_\varepsilon}\hat{\varphi}'\left(x',\dfrac{x}{r_\varepsilon}\right)\zeta\left(\dfrac{x_3}{\varepsilon}\right)\\ \varphi_{\varepsilon,3}(x)=\qquad\dfrac{\delta_\varepsilon}{\lambda\varepsilon r_\varepsilon}\hat{\varphi}_3\left(x',\dfrac{x}{r_\varepsilon}\right)\zeta\left(\dfrac{x_3}{\varepsilon}\right)-\dfrac{\delta_\varepsilon^2}{\lambda\varepsilon r_\varepsilon^2}\hat{\varphi}'\left(x',\dfrac{x}{r_\varepsilon}\right)\nabla\Psi\left(\dfrac{x'}{r_\varepsilon}\right)\zeta\left(\dfrac{x_3}{T_\varepsilon}\right).\end{cases}$$

Using properties (4.104) of $\tilde{\varphi}'$ and $\hat{\varphi}$, it is not difficult to check that, for ε small enough,

$$\varphi_\varepsilon=0 \text{ on } \partial\Omega_\varepsilon\setminus\Gamma_\varepsilon,\quad \varphi_\varepsilon\nu=0 \text{ on } \Gamma_\varepsilon,$$

which shows us φ_ε is a suitable test function for (4.102).

Taking into account that $\tilde{\varphi}'$, $\hat{\varphi}$ and Ψ and their derivatives are bounded, that $D_z\hat{\varphi}=0$ a.e in $\{z_3>M\}$ and properties (4.105) and (4.106), we have

$$D\varphi_\varepsilon(x)=\frac{1}{\varepsilon^2}\sum_{i=1}^{2}\partial_{y_3}\tilde{\varphi}_i\left(x',\frac{x_3}{\varepsilon}\right)e_i\otimes e_3+\frac{\delta_\varepsilon}{\lambda\varepsilon r_\varepsilon^2}D_z\hat{\varphi}\left(x',\frac{x}{r_\varepsilon}\right)+h_\varepsilon(x),$$

with $h_\varepsilon\in C^0(\bar{\Omega}_\varepsilon)^{3\times 3}$ satisfying

$$\varepsilon^3\int_{\Omega_\varepsilon}|h_\varepsilon|^2dx\le C\varepsilon^3\left(\frac{\delta_\varepsilon^2}{\varepsilon r_\varepsilon^2}+\frac{\delta_\varepsilon^2}{\varepsilon^3 r_\varepsilon^2}+\frac{\delta_\varepsilon^4 T_\varepsilon}{\varepsilon^2 r_\varepsilon^6}+\frac{\delta_\varepsilon^4}{\varepsilon^2 r_\varepsilon^2 T_\varepsilon}\right)=O_\varepsilon,$$

which, by (4.15) and (4.65) easily gives

$$\begin{aligned}\int_{\Omega_\varepsilon}Du_\varepsilon(x):D\varphi_\varepsilon(x)\,dx&=\frac{1}{\varepsilon^2}\int_{\Omega_\varepsilon}\partial_{y_3}u'_\varepsilon(x)\partial_{y_3}\tilde{\varphi}'\left(x',\frac{x_3}{\varepsilon}\right)\,dx\\&\quad+\frac{\delta_\varepsilon}{\lambda\varepsilon r_\varepsilon^2}\int_{\Omega_\varepsilon}Du_\varepsilon(x):D_z\hat{\varphi}\left(x',\frac{x}{r_\varepsilon}\right)\,dx+O_\varepsilon,\\ \int_{\Omega_\varepsilon}p_\varepsilon^0(x)\,\mathrm{div}\varphi_\varepsilon(x)\,dx&=\frac{\delta_\varepsilon}{\lambda\varepsilon r_\varepsilon^2}\int_{\Omega_\varepsilon}p_\varepsilon^0(x)\,\mathrm{div}_z\hat{\varphi}\left(x',\frac{x}{r_\varepsilon}\right)\,dx+O_\varepsilon.\end{aligned} \tag{4.107}$$

Using again that $\hat{\varphi}$ and $\nabla\Psi$ are bounded, thanks to (4.105), we obtain

$$\varepsilon\int_{\Omega_\varepsilon}\left(\left|\frac{\delta_\varepsilon}{\varepsilon r_\varepsilon}\hat{\varphi}\left(x',\frac{x_3}{r_\varepsilon}\right)\zeta\left(\frac{x_3}{\varepsilon}\right)\right|^2+\left|\frac{\delta_\varepsilon^2}{\varepsilon r_\varepsilon^2}\hat{\varphi}'\left(x',\frac{x_3}{r_\varepsilon}\right)\nabla\Psi\left(\frac{x'}{r_\varepsilon}\right)\zeta\left(\frac{x_3}{T_\varepsilon}\right)\right|^2\right)dx\le\frac{\delta_\varepsilon^2}{r_\varepsilon^2}+\frac{\delta_\varepsilon^4 T_\varepsilon}{r_\varepsilon^4\varepsilon}=O_\varepsilon,$$

which, by (4.15) and (4.65), and taking into account that (4.11) gives

$$\int_{\Omega_\varepsilon} |f_\varepsilon|^2 dx \le \varepsilon \int_{\omega\times(-1,1)} |\tilde{f}|^2 dy \le C\varepsilon,$$

proves

$$\begin{aligned}
&\int_{\Omega_\varepsilon} f_\varepsilon(x)\varphi_\varepsilon(x)\,dx = \frac{1}{\varepsilon}\int_{\Omega_\varepsilon} \tilde{f}'\left(x', \frac{x_3}{\varepsilon}\right)\tilde{\varphi}'\left(x', \frac{x_3}{\varepsilon}\right)\,dx + O_\varepsilon,\\
&\frac{\gamma}{\varepsilon}\int_{\Gamma_\varepsilon} u_\varepsilon\varphi_\varepsilon d\sigma = \frac{\gamma}{\varepsilon^2}\int_\omega u'_\varepsilon(x', -\delta_\varepsilon\Psi(\frac{x'}{r_\varepsilon}))\tilde{\varphi}'(x', -\frac{\delta_\varepsilon}{\varepsilon}\Psi(\frac{x'}{r_\varepsilon}))\sqrt{1+\frac{\delta_\varepsilon^2}{r_\varepsilon^2}|\nabla\Psi(\frac{x'}{r_\varepsilon}))|^2}dx' + O_\varepsilon, \quad (4.108)\\
&\int_{\Omega_\varepsilon} \nabla_{x'}p^1_\varepsilon(x')\varphi_\varepsilon(x)\,dx = \frac{1}{\varepsilon}\int_{\Omega_\varepsilon}\nabla_{x'}p^1_\varepsilon(x')\tilde{\varphi}'\left(x', \frac{x_3}{\varepsilon}\right)\,dx + O_\varepsilon.
\end{aligned}$$

Reasoning as in (4.103) but using now $\|\varphi_\varepsilon\|_{L^\infty(\Omega_\varepsilon)^3} \le C/\varepsilon$, we get

$$\left|\int_{\Omega_\varepsilon}(u_\varepsilon\cdot\nabla)u_\varepsilon\varphi_\varepsilon\,dx\right| \le \left(\int_{\Omega_\varepsilon}|u_\varepsilon|^4 dx\right)^{\frac14}\left(\int_{\Omega_\varepsilon}|Du_\varepsilon|^2 dx\right)^{\frac12}\left(\int_{\Omega_\varepsilon}|\varphi_\varepsilon|^4 dx\right)^{\frac14} \le C\varepsilon^{\frac73}. \quad (4.109)$$

Taking φ_ε as test function in (4.102), from (4.107), (4.108) and (4.109) we obtain

$$\begin{aligned}
&\frac{\mu}{\varepsilon^2}\int_{\Omega_\varepsilon}\partial_{y_3}u'_\varepsilon(x)\partial_{y_3}\tilde{\varphi}'\left(x', \frac{x_3}{\varepsilon}\right)\,dx + \frac{\mu\delta_\varepsilon}{\lambda\varepsilon r_\varepsilon^2}\int_{\Omega_\varepsilon} Du_\varepsilon(x) : D_z\hat{\varphi}\left(x', \frac{x}{r_\varepsilon}\right)\,dx\\
&+\frac{1}{\varepsilon}\int_{\Omega_\varepsilon}\nabla_{x'}p^1_\varepsilon(x')\tilde{\varphi}'\left(x', \frac{x_3}{\varepsilon}\right)\,dx - \frac{\delta_\varepsilon}{\lambda\varepsilon r_\varepsilon^2}\int_{\Omega_\varepsilon} p^0_\varepsilon(x)\,\mathrm{div}_z\hat{\varphi}\left(x', \frac{x}{r_\varepsilon}\right)\,dx\\
&+\frac{\gamma}{\varepsilon^2}\int_\omega u'_\varepsilon(x', -\delta_\varepsilon\Psi(\frac{x'}{r_\varepsilon}))\tilde{\varphi}'(x', -\frac{\delta_\varepsilon}{\varepsilon}\Psi(\frac{x'}{r_\varepsilon}))\sqrt{1+\frac{\delta_\varepsilon^2}{r_\varepsilon^2}|\nabla\Psi(\frac{x'}{r_\varepsilon}))|^2}dx'\\
&= \frac{1}{\varepsilon}\int_{\Omega_\varepsilon}\tilde{f}'\left(x', \frac{x_3}{\varepsilon}\right)\tilde{\varphi}'\left(x', \frac{x_3}{\varepsilon}\right)\,dx + O_\varepsilon.
\end{aligned}$$

Using the change of variables (4.16) in the terms with $\tilde{\varphi}'$ and the change (4.82) in the terms with $\hat{\varphi}$, last equality provides

$$
\begin{aligned}
&\frac{\mu}{\varepsilon^2}\int_\Omega \partial_{y_3}\tilde{u}'_\varepsilon(y)\partial_{y_3}\tilde{\varphi}'(y)\,dy + \frac{\mu\delta_\varepsilon}{\lambda\varepsilon r_\varepsilon^2}\int_\omega\int_{\hat{Q}_\varepsilon} D_z\hat{u}_\varepsilon(x',z) : D_z\hat{\varphi}(x',z)\,dzdx' \\
&+\frac{1}{\varepsilon}\int_{\tilde{\Omega}_\varepsilon} \nabla_{y'}\tilde{p}^1_\varepsilon(y')\tilde{\varphi}'(y)\,dy - \frac{\delta_\varepsilon}{\lambda\varepsilon r_\varepsilon}\int_\omega\int_{\hat{Q}_\varepsilon}\hat{p}^0_\varepsilon(x',z)\,\mathrm{div}\hat{\varphi}(x',z)\,dzdx' \\
&+\frac{\gamma}{\varepsilon^2}\int_\omega (\tilde{u}'_\varepsilon\tilde{\varphi}')\left(y', -\frac{\delta_\varepsilon}{\varepsilon}\Psi\left(\frac{y'}{r_\varepsilon}\right)\right)\sqrt{1+\frac{\delta_\varepsilon^2}{r_\varepsilon^2}\left|\nabla\Psi\left(\frac{y'}{r_\varepsilon}\right)\right|^2}dx' \\
&= \int_{\tilde{\Omega}_\varepsilon}\tilde{f}'(y)\tilde{\varphi}'(y)\,dy + O_\varepsilon.
\end{aligned}
\tag{4.110}
$$

By (4.69), the compact imbedding of $H^1(\Omega)$ into $L^2(\Gamma)$ and the inequality

$$
\int_\omega\left|\frac{1}{\varepsilon^2}\tilde{u}_\varepsilon\left(y', -\frac{\delta_\varepsilon}{\varepsilon}\Psi\left(\frac{y'}{r_\varepsilon}\right)\right) - \frac{1}{\varepsilon^2}\tilde{u}_\varepsilon(y',0)\right|^2 dy' \le C\frac{\delta_\varepsilon}{\varepsilon}\int_{\tilde{\Omega}_\varepsilon}|\frac{1}{\varepsilon^2}\partial_{y_3}\tilde{u}_\varepsilon|^2 dy, \tag{4.111}
$$

we have

$$
\frac{1}{\varepsilon^2}\tilde{u}'_\varepsilon\left(y', -\frac{\delta_\varepsilon}{\varepsilon}\Psi\left(\frac{y'}{r_\varepsilon}\right)\right) \longrightarrow \tilde{u}'(y',0) \quad \text{in } L^2(\omega)^2. \tag{4.112}
$$

Then, taking into account that by definition of λ and (4.2) we have

$$
\lim_{\varepsilon\to 0}\frac{\frac{\delta_\varepsilon}{\lambda\varepsilon r_\varepsilon^2}}{\frac{1}{\varepsilon\sqrt{\varepsilon r_\varepsilon}}} = \lim_{\varepsilon\to 0}\frac{\frac{\delta_\varepsilon}{\lambda\varepsilon r_\varepsilon}}{\frac{\sqrt{r_\varepsilon}}{\varepsilon^{3/2}}} = 1, \qquad \int_{\tilde{\Omega}_\varepsilon\setminus\Omega}|\tilde{\varphi}'|^2 dy \le C\frac{\delta_\varepsilon}{\varepsilon} = O_\varepsilon,
$$

and using (4.69), (4.80), (4.85), (4.99) and (4.112) we pass to the limit in (4.110) and obtain that $\tilde{u}'$, $\tilde{p}^1$, $\hat{p}^0$ and $\hat{u}$ satisfy

$$
\begin{aligned}
&\mu\int_\Omega \partial_{y_3}\tilde{u}'(y)\partial_{y_3}\tilde{\varphi}'(y)\,dy + \mu\int_\omega\int_{\hat{Q}} D_z\hat{u}(x',z) : D_z\hat{\varphi}(x',z)\,dzdx' \\
&+\int_\Omega \nabla_{y'}\tilde{p}^1(y')\tilde{\varphi}'(y)\,dy - \int_\omega\int_{\hat{Q}}\hat{p}^0(x',z)\,\mathrm{div}_z\hat{\varphi}(x',z)\,dzdx' \\
&+\gamma\int_\Gamma \tilde{u}'\tilde{\varphi}'d\sigma = \int_\Omega \tilde{f}'(y)\tilde{\varphi}'(y)\,dy,
\end{aligned}
\tag{4.113}
$$

for every $\tilde{\varphi}' \in C^1_c(\omega\times(-1,1))^2$, $\hat{\varphi} \in C^1_c(\omega; C^1_\sharp(\hat{Q})^3)$ with $D_z\hat{\varphi}(x',z) = 0$ a.e. in $\{z_3 > M\}$ for some constant $M > 0$, and such that (4.104) is satisfied. By density, this equality holds true for every $\tilde{\varphi}' \in H^1(\Omega)^2$ and every $\hat{\varphi} \in L^2(\omega; \mathcal{V}^3)$ such that

$$
\tilde{\varphi}' = 0 \ \text{ on } \partial\Omega\setminus\Gamma, \qquad \lambda\nabla\Psi(z')\tilde{\varphi}'(x',0) + \hat{\varphi}_3(x',z',0) = 0 \ \text{ a.e. } (x',z') \in \omega\times Z'.
$$

Let us now obtain a problem for $\tilde u'$ and $\tilde p^1$ eliminating $\hat u$ and $\hat p^0$ in (4.113). For this purpose, we take $\tilde\varphi' = 0$ in (4.113) which proves that $(\hat u, \hat p^0)$ (extended by periodicity to $\omega\times\mathbb{R}^2\times\mathbb{R}^+$) is a solution of

$$\begin{cases} -\mu\Delta_z\widehat{u} + \nabla_z\widehat{p^0} = 0 \;\text{ in } \mathbb{R}^2\times\mathbb{R}^+ \\ \text{div}_z\widehat{u} = 0 \;\text{ in } \mathbb{R}^2\times\mathbb{R}^+ \\ (\widehat{u},\widehat{p^0}) \in \mathcal{V}^3\times L^2_\sharp(\widehat{Q}) \\ \widehat{u}_3(x',z',0) = -\lambda\nabla\Psi(z')\tilde u'(x',0) \;\text{ on } \mathbb{R}^2\times\{0\} \\ \partial_{z_3}\widehat{u}' = 0 \;\text{ on } \mathbb{R}^2\times\{0\}, \end{cases} \tag{4.114}$$

a.e. x' in ω. Defining $(\widehat{\phi}^i,\widehat{q}^i)$, $i=1,2$, by (4.25), we deduce by linearity and uniqueness

$$\begin{aligned} &D_z\widehat{u}(x',z) = -\lambda(u_1(x',0)D_z\widehat{\phi}^1(z) + u_2(x',0)D_z\widehat{\phi}^2(z)) \;\text{ a.e. in } \mathbb{R}^2\times\mathbb{R}^+, \\ &\widehat{p^0}(x',z) = \lambda(u_1(x',0)\widehat{q}^1(z) + u_2(x',0)\widehat{q}^2(z)) \;\text{ a.e. in } \mathbb{R}^2\times\mathbb{R}^+. \end{aligned} \tag{4.115}$$

Now, for $\tilde\varphi'\in H^1(\Omega)^2$, with $\tilde\varphi'=0$ on $\partial\Omega\setminus\Gamma$, we take $\tilde\varphi'$ and $\hat\varphi(x',z) = -\lambda\big(\tilde\varphi_1(x',0)\widehat{\phi}^1(z) + \tilde\varphi_2(x',0)\widehat{\phi}^2(z)\big)$, as test functions in (4.113), which by (4.115) gives

$$\begin{aligned} &\mu\int_\Omega \partial_{y_3}\tilde u'\partial_{y_3}\tilde\varphi' dy + \int_\Omega \nabla_{y'}\tilde p^1\;\tilde\varphi' dy + \lambda^2\int_\omega R\tilde u'(y',0)\tilde\varphi'(y',0)dy' \\ &+\int_\Gamma \gamma\tilde u'\tilde\varphi'\,d\sigma = \int_\Omega \tilde f'\tilde\varphi' dy. \end{aligned} \tag{4.116}$$

By the arbitrariness of $\tilde\varphi'$, this together with (4.68) prove that $\tilde u'$, $\tilde w$ and $\tilde p = \tilde p^1$ is a solution of (4.21) and (4.27).

Step 3. Case $\lambda = 0$.

As in the previous step, we consider $\tilde\varphi'\in C^1_c(\omega\times(-1,1))^2$, with $\tilde\varphi'(x',x_3) = \tilde\varphi'(x',0)$ if $x_3\le 0$. Then, for $\zeta\in C^\infty(\mathbb{R})$ satisfying (4.105), we define $\varphi_\varepsilon\in H^1(\Omega_\varepsilon)^3$ by

$$\varphi'_\varepsilon(x) = \frac{1}{\varepsilon}\tilde\varphi'\left(x',\frac{x_3}{\varepsilon}\right) \qquad \varphi_{\varepsilon,3}(x) = -\frac{\delta_\varepsilon}{\varepsilon r_\varepsilon}\,\zeta\left(\frac{x_3}{r_\varepsilon}\right)\tilde\varphi'\left(x',0\right)\nabla\Psi\left(\frac{x'}{r_\varepsilon}\right).$$

For every $\varepsilon>0$, the function φ_ε satisfies $\varphi_\varepsilon = 0$ on $\partial\Omega_\varepsilon\setminus\Gamma_\varepsilon$, $\varphi_\varepsilon\nu = 0$ on Γ_ε. Then, taking φ_ε as test function in (4.102), passing to the limit, using that $\lambda = 0$ implies

$$\lim_{\varepsilon\to0}\left(\varepsilon^3\int_{\Omega_\varepsilon}\left|D\varphi_\varepsilon(x) - \sum_{i=1}^2\partial_{y_3}\tilde\varphi'_i\left(x',\frac{x_3}{\varepsilon}\right)e_i\otimes e_3\right|^2dx\right) = 0, \quad \lim_{\varepsilon\to0}\left(\varepsilon\int_{\Omega_\varepsilon}|\varphi_{\varepsilon,3}(x)|^2dx\right) = 0,$$

and reasoning by density, we get $\tilde{u}'$, $\tilde{p}$ satisfy

$$\mu \int_\Omega \partial_{y_3}\tilde{u}'\partial_{y_3}\tilde{\varphi}' dy + \int_\Omega \nabla_{y'}\tilde{p}^1 \ \tilde{\varphi}' dy + \int_\Gamma \gamma v' \tilde{\varphi}' \, d\sigma = \int_\Omega \tilde{f}' \tilde{\varphi}' dy, \tag{4.117}$$

for every $\tilde{\varphi}' \in H^1(\Omega)^2$ with $\tilde{\varphi}' = 0$ on $\partial\Omega \setminus \Gamma$, or equivalently that $\tilde{u}'$, $\tilde{p} = \tilde{p}^1$ satisfy (4.21) and (4.28).

Step 4. Case $\lambda = +\infty$.

We now consider $\tilde{\varphi}' \in C_c^\infty(\omega \times (-1,1))^2$ such that $\tilde{\varphi}'(x', x_3) = \tilde{\varphi}'(x', 0)$ if $x_3 \leq 0$ and

$$\tilde{\varphi}'(x', 0)\nabla\Psi\,(z') = 0 \ \text{ a.e. } (x', z') \in \omega \times Z'. \tag{4.118}$$

Observe that this choice of $\tilde{\varphi}$ implies that φ_ε defined by

$$\varphi_\varepsilon'(x) = \frac{1}{\varepsilon}\tilde{\varphi}'\left(x', \frac{x_3}{\varepsilon}\right) \qquad \varphi_{\varepsilon,3}(x) = 0,$$

satisfies $\varphi_\varepsilon = 0$ on $\partial\Omega_\varepsilon \setminus \Gamma_\varepsilon$, $\varphi_\varepsilon \nu = 0$ on Γ_ε. Taking φ_ε as test function in (4.102), passing to the limit and reasoning by density, we get $\tilde{u}'$, $\tilde{p}^1$ satisfy (4.117) for every $\tilde{\varphi}' \in H^1(\Omega)^2$ satisfying $\tilde{\varphi}' = 0$ on $\partial\Omega \setminus \Gamma$ and (4.118). It proves that $\tilde{u}'$, $\tilde{p} = \tilde{p}^1$ is a solution of (4.21) and (4.24).

□

Proof of Theorem 4.7.

Step 1. From (4.2), (4.15), $u_\varepsilon = 0$ on $\omega \times \{\varepsilon\}$ and Hölder's inequality, we get

$$\int_{\Omega_\varepsilon^-} |u_\varepsilon|^2 dx \leq C\varepsilon\delta_\varepsilon \int_{\Omega_\varepsilon} |\partial_{x_3} u_\varepsilon|^2 dx \leq C\varepsilon^4 \delta_\varepsilon. \tag{4.119}$$

On the other hand, applying the change of variables (4.16), the Rellich-Kondrachov theorem and (4.18), we obtain

$$\lim_{\varepsilon \to 0} \frac{1}{\varepsilon^5} \int_{\Omega_\varepsilon^+} \left(|u_\varepsilon'(x) - \varepsilon^2 \tilde{u}'(x', \frac{x_3}{\varepsilon})|^2 + |u_{\varepsilon,3}(x)|^2 \right) dx = 0.$$

This inequality and (4.119) prove that (4.39) holds for every value of $\lambda \in [0, +\infty]$.

Step 2. We consider $\varphi_\varepsilon \in H^1(\Omega_\varepsilon)^3$ such that

$$\varphi_\varepsilon = 0 \text{ on } \omega \times \{\varepsilon\}, \quad \varphi_\varepsilon \nu = 0 \text{ on } \Gamma_\varepsilon, \quad \fint_{\Omega_\varepsilon} |D\varphi_\varepsilon|^2 dx \leq C\varepsilon^2, \quad \fint_{\Omega_\varepsilon} |\text{div}\varphi_\varepsilon|^2 dx \leq C\varepsilon^4, \tag{4.120}$$

and there exists $\tilde{\varphi}' \in H^1(0,1;L^2(\omega))^2$, $\tilde{\rho} \in H^2(0,1;H^{-1}(\omega))$, $\tilde{\xi} \in L^2(\Omega)$ satisfying

$$\begin{gathered}\tilde{\varphi}'(1) = 0 \text{ in } L^2(\omega), \quad \tilde{\rho}(0) = \tilde{\rho}(1) = 0 \text{ in } H^{-1}(\omega),\\ \mathrm{div}_{y'}(\tilde{\varphi}') + \partial_{y_3}\tilde{\rho} = \tilde{\xi} \text{ in } H^1(0,1;H^{-1}(\omega)),\end{gathered} \tag{4.121}$$

and the following convergences hold

$$\frac{\tilde{\varphi}_\varepsilon}{\varepsilon} \rightharpoonup 0 \text{ in } H^1(\Omega)^3,\ \frac{\tilde{\varphi}_\varepsilon}{\varepsilon^2} \rightharpoonup (\tilde{\varphi}',0) \text{ in } H^1(0,1;L^2(\omega))^3,\ \frac{\tilde{\varphi}_{\varepsilon,3}}{\varepsilon^3} \rightharpoonup \tilde{\rho} \text{ in } H^2(0,1;H^{-1}(\omega)), \tag{4.122}$$

$$\frac{1}{\varepsilon^2}\mathrm{div}_{y'}(\tilde{\varphi}'_\varepsilon) + \frac{1}{\varepsilon^3}\partial_{y_3}\tilde{\varphi}_{\varepsilon,3} \rightharpoonup \tilde{\xi} \text{ in } L^2(\Omega), \tag{4.123}$$

where $\tilde{\varphi}_\varepsilon$ is defined from φ_ε using the change (4.16), i.e. $\tilde{\varphi}_\varepsilon(y) = \varphi_\varepsilon(y', \varepsilon y_3)$ a.e. $y \in \Omega$.

Let us prove that for any $\eta \in C_c^1(\omega)$, we have

$$\begin{aligned}&\lim_{\varepsilon\to 0}\left(\frac{\mu}{\varepsilon^3}\int_{\Omega_\varepsilon} Du_\varepsilon : D\varphi_\varepsilon \eta\, dx - \frac{1}{\varepsilon^3}\int_{\Omega_\varepsilon} p_\varepsilon\, \mathrm{div}_{x'}\varphi_\varepsilon \eta\, dx\right)\\ &= \mu\int_\Omega \partial_{y_3}\tilde{u}'\partial_{y_3}\tilde{\varphi}'\eta\, dy - \int_\Omega \tilde{p}\ \tilde{\xi}\ \eta\, dy, \qquad \text{if } \lambda = 0, +\infty,\end{aligned} \tag{4.124}$$

$$\begin{aligned}&\lim_{\varepsilon\to 0}\left(\frac{\mu}{\varepsilon^3}\int_{\Omega_\varepsilon} Du_\varepsilon : D\varphi_\varepsilon \eta\, dx - \frac{1}{\varepsilon^3}\int_{\Omega_\varepsilon} p_\varepsilon\, \mathrm{div}_{x'}\varphi_\varepsilon \eta\, dx\right)\\ &= \mu\int_\Omega \partial_{y_3}\tilde{u}'\partial_{y_3}\tilde{\varphi}'\eta\, dy - \int_\Omega \tilde{p}\ \tilde{\xi}\ \eta\, dy + \lambda^2\int_\Gamma R\tilde{u}'\tilde{\varphi}'\eta\, d\sigma, \qquad \text{if } \lambda \in (0, +\infty).\end{aligned} \tag{4.125}$$

For this purpose, given $\eta \in C_c^1(\omega)$, we take $\varphi_\varepsilon\eta/\varepsilon^3$ as test function in (4.10), and considering the functions p_ε^1, p_ε^0 given by Corollary 4.13 , this gives

$$\begin{aligned}&\frac{\mu}{\varepsilon^3}\int_{\Omega_\varepsilon} Du_\varepsilon(x) : D(\varphi_\varepsilon\eta)\, dx - \frac{1}{\varepsilon^3}\int_{\Omega_\varepsilon} p_\varepsilon^1 \mathrm{div}\,(\varphi_\varepsilon\eta)\, dx - \frac{1}{\varepsilon^3}\int_{\Omega_\varepsilon} p_\varepsilon^0 \mathrm{div}(\varphi_\varepsilon\eta)\, dx\\ &+\frac{1}{\varepsilon^3}\int_{\Omega_\varepsilon}(u_\varepsilon\cdot\nabla)u_\varepsilon\varphi_\varepsilon\eta\, dx + \frac{\gamma}{\varepsilon^4}\int_{\Gamma_\varepsilon} u_\varepsilon\varphi_\varepsilon\eta\, d\sigma = \frac{1}{\varepsilon^3}\int_{\Omega_\varepsilon} f_\varepsilon\varphi_\varepsilon\eta\, dx.\end{aligned} \tag{4.126}$$

Let us pass to the limit in each term of this equality.

The change of variables (4.16) and (4.122) gives

$$\frac{1}{\varepsilon^3}\int_{\Omega_\varepsilon^+} f_\varepsilon\varphi_\varepsilon\eta\, dx = \int_\Omega \tilde{f}\frac{\tilde{\varphi}_\varepsilon}{\varepsilon^2}\eta\, dy = \int_\Omega \tilde{f}'\tilde{\varphi}'\eta\, dy + O_\varepsilon. \tag{4.127}$$

On the other hand, since η is bounded and thanks to (4.120) and Poincaré's inequality, we have

$$\frac{1}{\varepsilon^3}\int_{\Omega_\varepsilon^-} f_\varepsilon\varphi_\varepsilon\eta\, dx = O_\varepsilon. \tag{4.128}$$

Convergences (4.18) and (4.122), the compact imbedding of $H^1(\Omega)$ into $L^2(\Gamma)$ and estimate (4.111) (which also holds true with $\tilde{u}_\varepsilon$ replaced by $\tilde{\varphi}_\varepsilon$) prove (4.112) and

$$\frac{1}{\varepsilon^2}\tilde{\varphi}'_\varepsilon\left(y', -\frac{\delta_\varepsilon}{\varepsilon}\Psi\left(\frac{y'}{r_\varepsilon}\right)\right) \longrightarrow \tilde{\varphi}'(y',0) \quad \text{in } L^2(\omega)^2.$$

From this, as $\Psi \in W^{2,\infty}(Z')$ and $\delta_\varepsilon/\varepsilon$ tends to zero, we deduce

$$\frac{\gamma}{\varepsilon^4}\int_{\Gamma_\varepsilon} u_\varepsilon\varphi_\varepsilon\eta\,d\sigma = \gamma\int_\omega\left(\frac{\tilde{u}_\varepsilon}{\varepsilon^2}\frac{\tilde{\varphi}_\varepsilon}{\varepsilon^2}\eta\right)\left(x', -\frac{\delta_\varepsilon}{\varepsilon}\Psi\left(\frac{x'}{r_\varepsilon}\right)\right)dx' + O_\varepsilon = \gamma\int_\Gamma \tilde{u}'\tilde{\varphi}'\eta\,d\sigma + O_\varepsilon. \quad (4.129)$$

Thanks to (4.15), (4.120) and (4.47), and using Hölder's inequality, we get

$$\left|\frac{1}{\varepsilon^3}\int_{\Omega_\varepsilon}(u_\varepsilon\cdot\nabla)u_\varepsilon\varphi_\varepsilon\eta\,dx\right| \le \frac{1}{\varepsilon^3}\|u_\varepsilon\|_{L^4(\Omega_\varepsilon)^3}\|Du_\varepsilon\|_{L^2(\Omega_\varepsilon)^{3\times3}}\|\varphi_\varepsilon\|_{L^4(\Omega_\varepsilon)^3} \le C\varepsilon^{\frac{5}{3}}. \quad (4.130)$$

Using Hölder's inequality, (4.47), (4.65), (4.120) and $\|\nabla\eta\|_{L^3(\Omega_\varepsilon)^3} \le C\varepsilon^{1/3}$, we obtain

$$\left|\frac{1}{\varepsilon^3}\int_{\Omega_\varepsilon} p^0_\varepsilon\varphi_\varepsilon\nabla_{x'}\eta\,dx\right| \le \frac{1}{\varepsilon^3}\|p^0_\varepsilon\|_{L^2(\Omega_\varepsilon)}\|\varphi_\varepsilon\|_{L^6(\Omega_\varepsilon)^3}\|\nabla_{x'}\eta\|_{L^3(\Omega_\varepsilon)^3} = O_\varepsilon. \quad (4.131)$$

Applying the change of variables (4.16), from (4.20) and (4.122), taking into account that $\tilde{p} = \tilde{p}^1$, it follows

$$\frac{1}{\varepsilon^3}\int_{\Omega^+_\varepsilon} p^1_\varepsilon\varphi'_\varepsilon\nabla_{x'}\eta\,dx = \frac{1}{\varepsilon^2}\int_{\Omega^+_\varepsilon}\tilde{p}^1_\varepsilon\tilde{\varphi}'_\varepsilon\nabla_{y'}\eta\,dy = \int_\Omega \tilde{p}\;\tilde{\varphi}'\nabla_{y'}\eta\,dy + O_\varepsilon. \quad (4.132)$$

Since $\varphi_\varepsilon = 0$ on $\omega\times\{\varepsilon\}$, then (4.15) and Poincaré inequality give $\|\varphi_\varepsilon\|_{L^2(\Omega^-_\varepsilon)} \le C\varepsilon^2\delta_\varepsilon^{1/2}$. This estimate together with (4.65) prove

$$\left|\frac{1}{\varepsilon^3}\int_{\Omega^-_\varepsilon} p^1_\varepsilon\varphi'_\varepsilon\nabla_{x'}\eta\,dx\right| \le \frac{C}{\varepsilon^3}\|p^1_\varepsilon\|_{L^2(\Omega_\varepsilon)}\|\varphi_\varepsilon\|_{L^2(\Omega^-_\varepsilon)} \le C\sqrt{\frac{\delta_\varepsilon}{\varepsilon}} = O_\varepsilon. \quad (4.133)$$

Finally, using again the change of variables (4.16), by (4.18), (4.122), we deduce

$$\frac{1}{\varepsilon^3}\int_{\Omega^+_\varepsilon} Du_\varepsilon : (\varphi_\varepsilon\otimes\nabla_{x'}\eta)\,dx = \int_\Omega \partial_{y_3}\tilde{u}' : (\tilde{\varphi}'\otimes\nabla_{y'}\eta)\,dy + O_\varepsilon, \quad (4.134)$$

whereas, by (4.15) and (4.47), we get

$$\left|\frac{1}{\varepsilon^3}\int_{\Omega^-_\varepsilon} Du_\varepsilon : (\varphi_\varepsilon\otimes\nabla_{x'}\eta)\,dx\right| \le \frac{1}{\varepsilon^3}\|Du_\varepsilon\|_{L^2(\Omega_\varepsilon)^{3\times3}}\|\varphi_\varepsilon\|_{L^6(\Omega_\varepsilon)^3}\|\nabla_{x'}\eta\|_{L^3(\Omega^-_\varepsilon)^3} \le \delta_\varepsilon^{\frac{1}{3}}. \quad (4.135)$$

By (4.127)-(4.135), we have then proved

$$\begin{aligned}&\lim_{\varepsilon\to 0}\left(\frac{\mu}{\varepsilon^3}\int_{\Omega_\varepsilon} Du_\varepsilon : D\varphi_\varepsilon\eta\,dx - \frac{1}{\varepsilon^3}\int_{\Omega_\varepsilon} p_\varepsilon\,\mathrm{div}_{x'}\varphi_\varepsilon\eta\,dx\right)\\&= -\mu\int_\Omega \partial_{y_3}\tilde{u}' : (\tilde{\varphi}'\otimes\nabla_{y'}\eta)dy - \gamma\int_\omega \tilde{u}'\tilde{\varphi}'\eta\,dx' + \int_\Omega \tilde{p}\;\tilde{\varphi}'\nabla_{y'}\eta\,dy + \int_\Omega \tilde{f}'\tilde{\varphi}'\eta\,dy.\end{aligned} \tag{4.136}$$

Since $\tilde{p}\in H^1(\Omega)$ is independent of y_3 and $\eta\in H_0^1(\omega)$, from (4.121), since $\tilde{\rho}(0)=\tilde{\rho}(1)=0$, we obtain

$$\begin{aligned}&\int_\Omega \tilde{p}\;\tilde{\varphi}'\nabla_{y'}\eta\,dy = -\int_0^1 <\mathrm{div}_{y'}\tilde{\varphi}',\tilde{p}\;\eta>_{H^{-1}(\omega),H_0^1(\omega)}\,dy_3 - \int_\Omega \nabla_{y'}\tilde{p}\;\tilde{\varphi}\eta\,dy\\&= -\int_\Omega \tilde{p}\;\tilde{\xi}\,\eta\,dy - \int_\Omega \nabla_{y'}\tilde{p}\;\tilde{\varphi}\eta\,dy.\end{aligned}$$

Using $\tilde{\varphi}'\eta$ as test function in the equation satisfied by $(\tilde{u}',\,\tilde{p})$ (Theorem 4.2), last equality and (4.136) prove (4.124)-(4.125).

Step 3. Let us prove (4.41), (4.43). Using $\varphi_\varepsilon = u_\varepsilon$ in Step 2 and taking into account that since $\mathrm{div}\,u_\varepsilon = 0$ in Ω_ε we have $\tilde{\xi}=0$, equalities (4.124) and (4.125) give, for every $\eta\in C_c^1(\omega)$, $\eta\geq 0$ in ω,

$$\lim_{\varepsilon\to 0}\frac{\mu}{\varepsilon^3}\int_{\Omega_\varepsilon} Du_\varepsilon : D\varphi_\varepsilon\eta\,dx = \mu\int_\Omega \partial_{y_3}\tilde{u}'\partial_{y_3}\tilde{\varphi}'\eta\,dy, \quad \text{if } \lambda = 0,+\infty, \tag{4.137}$$

$$\lim_{\varepsilon\to 0}\frac{\mu}{\varepsilon^3}\int_{\Omega_\varepsilon} Du_\varepsilon : D\varphi_\varepsilon\eta\,dx = \mu\int_\Omega \partial_{y_3}\tilde{u}'\partial_{y_3}\tilde{\varphi}'\eta\,dy + \lambda^2\int_\Gamma R\tilde{u}'\tilde{\varphi}'\eta\,d\sigma, \quad \text{if } \lambda\in(0,+\infty). \tag{4.138}$$

Since $\tilde{u}_\varepsilon/\varepsilon$ converges weakly to zero in $H^1(\Omega)^3$, $\tilde{u}_\varepsilon/\varepsilon^2$ converges to $(\tilde{u}',0)$ in $H^1(0,1;L^2(\omega)^3)$, equality (4.137) proves (4.41).

In order to prove (4.43), we take $s_\varepsilon > 0$ such that

$$\lim_{\varepsilon\to 0}\frac{s_\varepsilon}{\varepsilon} = 0, \qquad \lim_{\varepsilon\to 0}\frac{s_\varepsilon}{r_\varepsilon} = +\infty. \tag{4.139}$$

Then we decompose

$$\begin{aligned}&\frac{1}{\varepsilon^3}\int_{\Omega_\varepsilon} |Du_\varepsilon|^2\eta\;dx = \frac{1}{\varepsilon^3}\int_{\Omega_\varepsilon^-} |Du_\varepsilon|^2\eta\;dx + \frac{1}{\varepsilon^3}\int_{\{x_3>s_\varepsilon\}} |Du_\varepsilon|^2\eta\,dx\\&+\frac{1}{\varepsilon^3}\int_{\{0<x_3<s_\varepsilon\}} |Du_\varepsilon|^2\eta\;dx.\end{aligned} \tag{4.140}$$

Let us estimate each term on the right-hand side of (4.140).

Clearly

$$\liminf_{\varepsilon\to 0} \frac{1}{\varepsilon^3} \int_{\Omega_\varepsilon^-} |Du_\varepsilon|^2 \eta \; dx \geq 0. \tag{4.141}$$

By (4.139), using that $\tilde{u}_\varepsilon/\varepsilon$ converges weakly to zero in $H^1(\Omega)^3$ and $\tilde{u}_\varepsilon/\varepsilon^2$ converges to $(\tilde{u}', 0)$ in $H^1(0, 1; L(\omega)^3)$, for the second term on the right of (4.140) we have for every $\tau > 0$

$$\liminf_{\varepsilon\to 0} \frac{1}{\varepsilon^3} \int_{\{x_3 > s_\varepsilon\}} |Du_\varepsilon|^2 \eta \, dx \geq \int_{\{y_3 > \tau\}} \left(\frac{1}{\varepsilon^2} |D_{y'} \tilde{u}_\varepsilon|^2 + \frac{1}{\varepsilon^4} |\partial_{y_3} \tilde{u}_\varepsilon|^2 \right) \eta \, dy \geq \int_{\{y_3 > \tau\}} |\partial_{y_3} \tilde{u}'|^2 \eta \, dy.$$

So we have

$$\liminf_{\varepsilon\to 0} \frac{1}{\varepsilon^3} \int_{\{x_3 > s_\varepsilon\}} |Du_\varepsilon|^2 \eta \, dx \geq \sup_{\tau > 0} \int_{\{y_3 > \tau\}} |\partial_{y_3} \tilde{u}'|^2 \eta \, dy = \int_\Omega |\partial_{y_3} \tilde{u}'|^2 \eta \, dy. \tag{4.142}$$

For the third term on the right of (4.140), we take $M > 0$ and $\varepsilon > 0$ small enough such that $M < \frac{s_\varepsilon}{r_\varepsilon}$. Defining $\hat{u}_\varepsilon$ by (4.81) and using the change of variables (4.82) and the uniform continuity of η, we get

$$\begin{aligned} \frac{1}{\varepsilon^3} \int_{\{0 < x_3 < s_\varepsilon\}} |Du_\varepsilon|^2 \eta \; dx &= \int_{\omega \times \widehat{Q}_{\frac{s_\varepsilon}{r_\varepsilon}}} |D_z \left(\frac{\hat{u}_\varepsilon}{\varepsilon \sqrt{\varepsilon r_\varepsilon}} \right)|^2 \eta \, dx' dz + O_\varepsilon \\ &\geq \int_{\omega \times \widehat{Q}_M} |D_z \left(\frac{\hat{u}_\varepsilon}{\varepsilon \sqrt{\varepsilon r_\varepsilon}} \right)|^2 \eta \, dx' dz + O_\varepsilon. \end{aligned} \tag{4.143}$$

On the other hand, by (4.85) we have that $D_z(\frac{\hat{u}_\varepsilon}{\varepsilon\sqrt{\varepsilon r_\varepsilon}})$ converges weakly to $D_z \hat{u}$, with $\hat{u}$ defined by (4.42), in $L^2(\omega \times \widehat{Q}_M)^{3\times 3}$, for every $M > 0$. Therefore

$$\begin{aligned} \liminf_{\varepsilon\to 0} \frac{1}{\varepsilon^3} \int_{\{0 < x_3 < s_\varepsilon\}} |Du_\varepsilon|^2 \eta \; dx &\geq \sup_{M > 0} \int_{\omega \times \widehat{Q}_M} |D_z \hat{u}|^2 \eta \, dx' dz \\ &= \int_{\omega \times \widehat{Q}} |D_z \hat{u}|^2 \eta \, dx' dz = \lambda^2 \int_\Gamma R \tilde{u}' \tilde{u}' \eta \, dx'. \end{aligned} \tag{4.144}$$

By (4.138), statements (4.140)-(4.144) imply

$$\lim_{\varepsilon\to 0} \frac{1}{\varepsilon^3} \int_{\Omega_\varepsilon^-} |Du_\varepsilon|^2 \eta \; dx = 0, \tag{4.145}$$

$$\lim_{\varepsilon\to 0} \frac{1}{\varepsilon^3} \int_{\{x_3 > s_\varepsilon\}} |Du_\varepsilon|^2 \eta \, dx = \int_\Omega |\partial_{y_3} \tilde{u}'|^2 \eta \, dy, \tag{4.146}$$

and

$$\lim_{\varepsilon\to0}\frac{1}{\varepsilon^3}\int_{\{0<x_3<s_\varepsilon\}}|Du_\varepsilon|^2\eta\,dx=\lim_{\varepsilon\to0}\int_{\omega\times\widehat{Q}_{\frac{s_\varepsilon}{r_\varepsilon}}}|D_z\left(\frac{\hat{u}_\varepsilon}{\varepsilon\sqrt{\varepsilon r_\varepsilon}}\right)|^2\eta\,dx'dz=\int_{\omega\times\widehat{Q}}|D_z\hat{u}|^2\eta\,dx'dz. \tag{4.147}$$

From (4.146), (4.147), the weak convergences of $\tilde{u}_\varepsilon/\varepsilon$ to zero in $H^1(\Omega)^3$, of $\tilde{u}_\varepsilon/\varepsilon^2$ to $(\tilde{u}',0)$ in $H^1(0,1;L^2(\omega)^3)$ and of $\frac{1}{\varepsilon\sqrt{\varepsilon r_\varepsilon}}D_z\hat{u}_\varepsilon$ to $D_z\hat{u}$ in $L^2(\omega_\rho\times\widehat{Q}_M)^{3\times3}$, for every $M>0$, we obtain

$$\lim_{\varepsilon\to0}\frac{1}{\varepsilon^3}\int_{\{x_3>s_\varepsilon\}}\left|Du_\varepsilon(x)-\varepsilon\sum_{i=1}^{2}\partial_{y_3}\tilde{u}'\left(x',\frac{x_3}{\varepsilon}\right)e_i\otimes e_3\right|^2\eta(x')\,dx=0, \tag{4.148}$$

$$\lim_{\varepsilon\to0}\int_{\omega\times\widehat{Q}_{\frac{s_\varepsilon}{r_\varepsilon}}}\left|D_z\left(\frac{\hat{u}_\varepsilon(x',z)}{\varepsilon\sqrt{\varepsilon r_\varepsilon}}-\hat{u}(x',z)\right)\right|^2\eta(x')\,dx'dz=0. \tag{4.149}$$

Therefore, taking $\rho>0$ such that $\eta(x')=0$ if $x'\notin\omega_\rho$ and using that $\hat{u}_\varepsilon(x',z)$ does not depend on x' in $C^{k'}_{r_\varepsilon}\times Z'$, for every $k'\in I_{\rho,\varepsilon}$, we get

$$\begin{aligned}&\int_{\{0<x_3<s_\varepsilon\}}\left|\frac{1}{\varepsilon^{\frac{3}{2}}}Du_\varepsilon(x)-⨍_{C_{r_\varepsilon}(x')}D_z\left(\frac{\hat{u}}{\sqrt{r_\varepsilon}}\right)(s',\frac{x}{r_\varepsilon})ds'\right|^2\eta(x')\,dx\\&=r_\varepsilon^3\sum_{k'}\int_{\widehat{Q}_{\frac{s_\varepsilon}{r_\varepsilon}}}\left|\frac{1}{r_\varepsilon^{5/2}}\int_{C^{k'}_{r_\varepsilon}}D_z\left(\frac{\hat{u}_\varepsilon(s',z)}{\varepsilon\sqrt{\varepsilon r_\varepsilon}}-\hat{u}(s',z)\right)ds'\right|^2\eta(x')dz+O_\varepsilon\\&\leq\int_{\omega\times\widehat{Q}_{\frac{s_\varepsilon}{r_\varepsilon}}}\left|\frac{1}{\varepsilon\sqrt{\varepsilon r_\varepsilon}}D_z\hat{u}_\varepsilon(x',z)-D_z\hat{u}(x',z)\right|^2\eta(x')\,dx'dz+O_\varepsilon=O_\varepsilon.\end{aligned} \tag{4.150}$$

By (4.139) we have

$$\begin{aligned}&\lim_{\varepsilon\to0}\frac{1}{\varepsilon}\int_{\{x_3<s_\varepsilon\}}\left|\sum_{i=1}^{2}\partial_{y_3}\tilde{u}'\left(x',\frac{x_3}{\varepsilon}\right)e_i\otimes e_3\right|^2\eta(x')\,dx=0\\&\lim_{\varepsilon\to0}\int_{\{x_3>s_\varepsilon\}}\left|\frac{1}{r_\varepsilon}⨍_{C_{r_\varepsilon}(x')}D_z\hat{u}(\zeta',\frac{x}{r_\varepsilon})d\zeta'\right|^2\eta(x')\,dx=0,\end{aligned}$$

and then, from (4.145), (4.148) and (4.150), we deduce (4.41) and (4.43).

Step 4. Let us now prove that (4.40) holds.

For every $\varepsilon>0$, by Proposition 4.10 (ii) there exists $\phi_\varepsilon\in H^1_0(\Omega_\varepsilon)^3$ satisfying

$$\operatorname{div}\phi_\varepsilon=p_\varepsilon=p^1_\varepsilon+p^0_\varepsilon\ \text{ in }\Omega_\varepsilon, \tag{4.151}$$

and, by (4.46) and (4.65),

$$\|\phi_\varepsilon\|_{H_0^1(\Omega_\varepsilon)^3} \le \frac{C}{\sqrt{\varepsilon}}, \quad \forall \varepsilon > 0. \tag{4.152}$$

Thus, taking into account that $\phi_\varepsilon = 0$ on Γ_ε, (4.20) and (4.151), from Lemma 4.14 applied to the sequence $\varepsilon^2\phi_\varepsilon$ we deduce that there exists $\tilde{\phi}' \in H^1(0,1;L^2(\omega)^2)$ and $\tilde{\theta} \in H^2(0,1;H^{-1}(\omega))$ satisfying

$$\tilde{\phi}'(0) = \tilde{\phi}'(1) = 0 \text{ in } L^2(\omega), \quad \tilde{\theta}(0) = \tilde{\theta}(1) = 0 \text{ in } H^{-1}(\omega),$$
$$\mathrm{div}_{y'}(\tilde{\phi}') + \partial_{y_3}\tilde{\theta} = \tilde{p} \text{ in } H^1(0,1;H^{-1}(\omega)),$$

and the following convergences hold

$$\begin{gathered}\varepsilon\tilde{\phi}_\varepsilon \rightharpoonup 0 \text{ in } H^1(\Omega)^3,\ \tilde{\phi}_\varepsilon \rightharpoonup (\tilde{\phi}', 0) \text{ in } H^1(0,1;L^2(\omega)^3),\ \frac{\tilde{\phi}_{\varepsilon,3}}{\varepsilon} \rightharpoonup \tilde{\theta} \text{ in } H^2(0,1;H^{-1}(\omega)),\\ \mathrm{div}_{y'}(\tilde{\phi}'_\varepsilon) + \frac{1}{\varepsilon}\partial_{y_3}\tilde{\phi}_{\varepsilon,3} \rightharpoonup \tilde{p} \text{ in } L^2(\Omega),\end{gathered} \tag{4.153}$$

where as usual $\tilde{\phi}_\varepsilon$ is defined from ϕ_ε using the change of variables (4.16), i.e. $\tilde{\phi}_\varepsilon(y) = \phi_\varepsilon(y', \varepsilon y_3)$ a.e. $y \in \Omega$. Taking $\varphi_\varepsilon = \varepsilon^2\phi_\varepsilon$ in Step 2, equalities (4.124) and (4.125) give for every $\eta \in C_c^1(\omega)$, $\eta \ge 0$ in ω,

$$\begin{aligned}&\lim_{\varepsilon\to 0}\left(\frac{\mu}{\varepsilon}\int_{\Omega_\varepsilon} Du_\varepsilon : D\phi_\varepsilon \eta\, dx - \frac{1}{\varepsilon}\int_{\Omega_\varepsilon}|p_\varepsilon|^2\eta\, dx\right)\\ &= \mu\int_\Omega \partial_{y_3}\tilde{u}'\partial_{y_3}\tilde{\phi}'\eta\, dy - \int_\Omega |\tilde{p}|^2\eta\, dy, \quad \forall\lambda \in [0, +\infty].\end{aligned} \tag{4.154}$$

If $\lambda = 0,\ +\infty$, then (4.41), (4.152) and (4.153) give

$$\lim_{\varepsilon\to 0}\frac{1}{\varepsilon}\int_{\Omega_\varepsilon} Du_\varepsilon : D\phi_\varepsilon\, \eta\, dx = \int_\Omega \partial_{y_3}v'\partial_{y_3}\phi'\, \eta\, dy,$$

thus, by (4.154), we deduce

$$\lim_{\varepsilon\to 0}\frac{1}{\varepsilon}\int_{\Omega_\varepsilon}|p_\varepsilon|^2\eta\, dy = \int_\Omega |\tilde{p}|^2\, \eta\, dy, \tag{4.155}$$

which, using the change of variables (4.16) and that $\tilde{p}_\varepsilon$ converges weakly to $\tilde{p}$ in $L^2(\Omega)$, proves (4.40).

If $\lambda \in (0, +\infty)$, we apply Lemma 4.18 to $\varepsilon^2\phi_\varepsilon$ which gives the existence of $\hat{\phi} \in L^2(\Omega; \mathcal{V}^3)$ such that, as $\tilde{\phi}' = 0$ on Γ, satisfies

$$\hat{\phi}_3(x', z', 0) = -\lambda\nabla\Psi(z')\tilde{\phi}'(x', 0) = 0 \quad \text{a.e. } (x', z') \in \omega \times Z', \tag{4.156}$$

and such that, up to a subsequence, $\widehat{\phi}_\varepsilon(x',z) = \phi_\varepsilon(r_\varepsilon\kappa(\frac{x'}{r_\varepsilon})+r_\varepsilon z', r_\varepsilon z_3)$ a.e. $(x',z') \in \omega_\rho \times \widehat{Z}_\varepsilon$ satisfies

$$\sqrt{\frac{\varepsilon}{r_\varepsilon}} D_z \widehat{\phi}_\varepsilon \rightharpoonup D_z \widehat{\phi} \text{ in } L^2(\omega_\rho \times \widehat{Q})^3, \quad \forall \rho, M > 0. \tag{4.157}$$

In particular, taking into account (4.99), (4.151) and

$$\int_{\omega_\rho \times \widehat{Q}_M} |\sqrt{\varepsilon r_\varepsilon} \widehat{p^1}_\varepsilon|^2 \, dx' \, dz \leq \sum_{k' \in I_{\rho,\varepsilon}} \varepsilon r_\varepsilon^3 \int_{\widehat{Q}_M} |p_\varepsilon^1(r_\varepsilon(k'+z'), r_\varepsilon z_3)|^2 \, dz$$

$$\leq \sum_{k' \in I_{\rho,\varepsilon}} \varepsilon \int_{\Omega^{k'}_{r_\varepsilon}} |p_\varepsilon^1(x)|^2 \, dx \leq \varepsilon \int_{\Omega_\varepsilon} |p_\varepsilon^1|^2 \, dx \leq C\varepsilon^2,$$

we deduce

$$\sqrt{\frac{\varepsilon}{r_\varepsilon}} \text{div}_z \widehat{\phi}_\varepsilon = \sqrt{\varepsilon r_\varepsilon} \widehat{p^1}_\varepsilon + \sqrt{\varepsilon r_\varepsilon} \widehat{p^0}_\varepsilon \to 0 \text{ in } L^2(\omega_\rho \times \widehat{Q})^3, \quad \forall \rho, M > 0,$$

which gives

$$\text{div}_z \widehat{\phi} = 0 \quad \text{in } \omega \times \hat{Q}. \tag{4.158}$$

Using (4.43), (4.152), (4.153), the change of variables (4.82), the uniform continuity of η and (4.157) we obtain

$$\lim_{\varepsilon \to 0} \frac{1}{\varepsilon} \int_{\Omega_\varepsilon} Du_\varepsilon : D\phi_\varepsilon \, \eta \, dx = \int_\Omega \partial_{y_3} \tilde{u}' \partial_{y_3} \tilde{\phi}' \, \eta \, dy$$

$$+ \lim_{\varepsilon \to 0} \int_{\Omega_\varepsilon} \sqrt{\frac{\varepsilon}{r_\varepsilon}} \fint_{C_{r_\varepsilon}(x')} D_z \widehat{u}(x', \frac{x}{r_\varepsilon}) ds' : D\phi_\varepsilon \, \eta \, dx = \int_\Omega \partial_{y_3} \tilde{u}' \partial_{y_3} \phi' \, \eta \, dy \tag{4.159}$$

$$+ \int_{\omega \times \widehat{Q}} D_z \widehat{u}(x',z) : D_z \widehat{\phi}(x',z) \eta(x') \, dx' dz,$$

but taking $\widehat{\phi}$ as test function in (4.114), thanks to (4.156) and (4.158) we deduce

$$\int_{\omega \times \widehat{Q}} D_z \widehat{u} : D_z \widehat{\phi} \, \eta \, dx' dz = \int_{\omega \times \widehat{Q}} \widehat{p^0} \, \text{div}_z \hat{\phi} \, \eta \, dx' dz = 0. \tag{4.160}$$

Then (4.154), (4.159) and (4.160) give

$$\lim_{\varepsilon \to 0} \frac{1}{\varepsilon} \int_{\Omega_\varepsilon} |p_\varepsilon|^2 \, \eta \, dx = \int_\Omega |\tilde{p}|^2 \, \eta \, dy,$$

which, by using the change of variables (4.16) and that $\tilde{p}_\varepsilon$ converges weakly to $\tilde{p}$ in $L^2(\Omega)$, proves (4.40).

□

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Chapter 5

The homogenization of elliptic partial differential systems on rugous domains with variable boundary conditions

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Abstract.

The present paper is devoted to study the asymptotic behavior of a sequence of elliptic systems posed in a sequence of rugous domains Ω_n. The solutions are assumed to belong to a vectorial space $V_n(x)$ depending on $x \in \overline{\Omega}_n$. This permits to consider several types of boundary conditions posed in variables sets of the boundary and in particular contains classical results for the homogenization of Dirichlet elliptic problems in varying domains.

5.1 Introduction

The goal of the present paper is to study the homogenization of a sequence of elliptic systems on rugous domains. Namely, we consider a sequence of Lipschitz open sets $\Omega_n \subset \mathbb{R}^N$, which are converging to a bounded Lipschitz open set $\Omega \subset \mathbb{R}$ in the following sense: For every

$\rho > 0$, there exists $n_0 \in \mathbb{N}$ such that for every $n \geq n_0$,

$$\{x \in \Omega : d(x, \partial\Omega) < \rho\} \subset \Omega_n \subset \left\{x \in \mathbb{R}^N : d(x, \overline{\Omega}) < \rho\right\}. \tag{5.1}$$

We also consider a tensor bounded measurable function A from an open set $\widetilde{\Omega}$ containing strictly Ω into $\mathcal{T}_{M\times N}$ (the space of linear aplications from the space of matrices $\mathcal{M}_{M\times N}$ into itself), and a sequence of functions V_n from $\overline{\Omega}_n$ into the set of linear subspaces of $\mathbb{R}^N$. We assume that there exists $\alpha > 0$, independently of n, such that

$$\alpha\|v\|^2_{H^1(\Omega_n)^M} \leq \int_{\Omega_n} ADv : Dv\, dx, \quad \forall\, v \in H^1(\Omega_n)^M, \text{ with } v(x) \in V(x) \text{ for q.e. } x \in \overline{\Omega}_n.$$

Then, for two bounded sequences f_n and G_n in $L^2(\Omega_n)^M$ and $L^2(\Omega_n)^{M\times N}$ respectively, which converge to some $f \in L^2(\Omega)^M$ and $G \in L^2(\Omega)^{M\times N}$ in the sense that (5.17) and (5.18) below are satisfied let us study the homogenization problem

$$\begin{cases} u_n \in V_n \ \text{ q.e. in } \overline{\Omega}_n \\ \displaystyle\int_{\Omega_n} ADu_n : Dv\, dx = \int_{\Omega_n} f_n \cdot v\, dx + \int_{\Omega_n} G_n : Dv\, dx, \ \forall\, v \in H^1(\Omega_n)^M, \ v \in V \text{ q.e. in } \overline{\Omega}_n, \end{cases} \tag{5.2}$$

Our main result shows the existence of a subsequence of n, still denoted by n, a Radon measure μ in $\overline{\Omega}$, a μ-measurable function $R : \overline{\Omega} \to \mathcal{M}_{M\times M}$ and an application V from $\overline{\Omega}$ into the set of linear subspaces on $\mathbb{R}^N$, satisfying

$$R\xi\cdot\xi \geq 0, \ |R\xi\cdot\eta| \leq \beta(R\xi\cdot\xi)^{\frac{1}{2}}(R\eta\cdot\eta)^{\frac{1}{2}}, \quad \forall\, \xi, \eta \in \mathbb{R}^N, \ \mu\text{-a.e. in } \overline{\Omega},$$

for some $\beta > 0$, and

$$\begin{gathered} \alpha\|v\|^2_{H^1(\Omega)^M} \leq \int_\Omega ADv : Dv\, dx + \int_{\overline{\Omega}} Rv\cdot v\, d\mu \\ \forall\, v \in H^1(\Omega)^M, \text{ with } v(x) \in V(x) \text{ for q.e. } x \in \overline{\Omega}, \end{gathered} \tag{5.3}$$

such that for every $\rho > 0$ the solutions of (5.2) converge weakly in $H^1(\{x \in \Omega : d(x, \partial\Omega) < \rho\})$ to the unique solution u of the variational problem

$$\begin{cases} u \in H^1(\Omega)^M, \ u \in V \text{ q.e. in } \overline{\Omega}, \ \displaystyle\int_{\overline{\Omega}} Ru\cdot u\, d\mu < +\infty \\ \displaystyle\int_\Omega ADu : Dv\, dx + \int_{\overline{\Omega}} Ru\cdot v\, d\mu = \int_\Omega f\cdot v\, dx + \int_\Omega G : Dv\, dx \\ \forall\, v \in H^1(\Omega)^M, \ v \in V \text{ q.e. in } \overline{\Omega}, \ \displaystyle\int_{\overline{\Omega}} Rv\cdot v\, d\mu < +\infty. \end{cases} \tag{5.4}$$

The subsequence of n, the measure μ and the applications R and V do not depend on f_n, G_n, f and G. More generally, we will show that problem (5.2) is stable by homogenization.

Assuming there exists a closed set C_n such that $V_n = \{0\}$ on C_n, $V_n = \mathbb{R}^M$ on $\Omega_n \setminus C_n$ and V_n arbitrary in $\partial\Omega_n \setminus C_n$, problem (5.2) can be written as

$$\begin{cases} -\mathrm{div}(ADu_n - G_n) = f_n \ \text{ in } \Omega_n \setminus C_n \\ u_n = 0 \text{ on } C_n \\ u_n \in V_n, \ (ADu_n - G_n)\nu \in V_n^{\perp} \ \text{ on } \partial\Omega_n \setminus C_n \end{cases} \tag{5.5}$$

with ν the outside normal vector to Ω_n on $\partial\Omega_n$. For $\Omega_n = \Omega$, and $V_n = \{0\}$ on $\partial\Omega$ problem (5.5) is the classical homogenization problem for linear elliptic equations in varying domains with Dirichlet conditions. In this case, the term $Ru\mu$ which appears in the limit equation is what, in the terminology of D. Cioranescu and F. Murat, is known as the strange term (see e.g. [7], [8], [10], [13], [15], [17], [18], [19], [20], for the homogenization of linear and nonlinear elliptic problems in varying domains with Dirichlet conditions).

Taking $C_n = \emptyset$, problem (5.5) permits to incorporate several boundary conditions. In this case it is simple to check that in (5.4), the measure μ is concentrated on $\partial\Omega$ and that $V = \mathbb{R}^M$ in Ω. So, (5.4) can be written (at least formally) as the following problem with a generalized Fourier's condition with by our main result is stable by homogenization

$$\begin{cases} -\mathrm{div}(ADu - G) = f \ \text{ in } \Omega \\ \displaystyle\int_{\partial\Omega} Ru \cdot u \, d\mu < +\infty, \qquad u \in V, \ (ADu - G)\nu + Ru\mu \in V^{\perp} \ \text{ on } \partial\Omega. \end{cases}$$

In particular, for $V_n = \mathbb{R}^M$ on Ω_n and V_n taking only the values $\{0\}$ and $\mathbb{R}^M$ on the boundary, Problem (5.5) corresponds to the homogenization on an elliptic problem in Ω_n where we impose a Dirichlet condition on a varying subset of the boundary and a Neuman condition on the rest. This problem has been studied in [5] and [6] assuming $\Omega_n = \Omega$.

One difference between the present work ant the references mentioned above is that here the ellipticity assumption (5.3) imposed to the operators is written in an integral form at the place of a pointwise one. This is more convenient for systems and in particular for the linear elasticity where the tensor only depends on the symmetric part of the derivative (see Theorem 5.2 below).

Introducing Ω_n at the place of Ω in Theorem allows us to work with rugous boundaries. In this sense we refer to [4] where it is studied the homogenization of the Stokes system (Navier-Stokes is also considered) with Navier's conditions on the boundary on rugous domains

$$\begin{cases} -\Delta u_n + u_n + \nabla p_n = f_n \ \text{ in } \Omega_n \\ \mathrm{div}\, u_n = 0 \ \text{ in } \Omega_n \\ u_n \cdot \nu = 0 \ \text{ on } \partial\Omega_n, \quad \dfrac{\partial u_n}{\partial \nu} = 0 \ \text{ on } \partial\Omega_n. \end{cases}$$

Remark that denoting by $T(x)$ the tangent space in a point x of the boundary of Ω_n, the equation for u_n can be written as (compare with (5.19))

$$\begin{cases} u_n \in H^1(\Omega)^N, \quad \operatorname{div} u_n = 0 \text{ on } \Omega_n, \quad u_n \in T \text{ q.e. on } \partial\Omega_n \\ \displaystyle\int_{\Omega_n} Du_n : Dv\,dx + \int_{\Omega_n} u_n \cdot v\,dx = \int_{\Omega_n} f_n \cdot v\,dx \\ \forall\, v \in H^1(\Omega_n)^N, \quad \operatorname{div} v = 0 \text{ on } \Omega_n, \quad v \in T \text{ q.e. on } \partial\Omega_n. \end{cases} \tag{5.6}$$

Assuming appropiate conditions on Ω_n it is proved similarly to (5.20) the existence of an application V from $\partial\Omega$ into the set of linear subspaces of $\mathbb{R}^N$, with $V(x) \subset T(x)$ for every $x \in \partial\Omega$, a measure μ on $\partial\Omega$, vanishing on the sets of zero capacity, and a μ-measure function $R : \partial\Omega \to \mathbb{R}^{N\times N}$ such that the limit problem of (5.6) is given by

$$\begin{cases} u \in H^1(\Omega)^N, \quad \operatorname{div} u = 0 \text{ on } \Omega, \quad u \in V \text{ q.e. on } \partial\Omega, \quad \displaystyle\int_{\partial\Omega} Ru \cdot u\,d\mu < +\infty \\ \displaystyle\int_{\Omega} Du : Dv\,dx + \int_{\Omega} u \cdot v\,dx + \int_{\partial\Omega} Ru \cdot v\,d\mu = \int_{\Omega} f \cdot v\,dx \\ \forall\, v \in H^1(\Omega)^N, \quad \operatorname{div} v = 0 \text{ on } \Omega, \quad v \in V \text{ q.e. on } \partial\Omega, \quad \displaystyle\int_{\partial\Omega} Rv \cdot v\,d\mu < +\infty. \end{cases}$$

Related results can be found in [1], [2], [3], [9], [1], [11], [12] and in particular some conditions on the geometry of Ω_n assuring that in the limit $V(x) = \{0\}$ for every $x \in \partial\Omega$. This permits to show that a for a sufficiently rugous boundary Navier's condition implies the usual adherence condition for viscous fluids, $u = 0$ on the boundary, and so mathematically justifies that due to the existence of microrugosities, a viscous fluid adheres to the boundary .

The results of [4] are based on an integral representation theorem which appears in [16]. Analogously the proof of our main result is based on a variant of this representation theorem which we will give in section 5.4, Theorem 5.4. We remark that the result which appears in [16] is valid for convex functionals and thus permits to work with nonlinear PDE. It is more useful when we apply Γ- convergence techniques in homogenization ([14]). The variant we present here assumes the functional quadratic and so it is only valid for linear PDE, but it does not assumes that the functional is convex and so the diffusion term of the PDE has not to be necessarily symmetric. It is more useful when we apply H-convergence techniques in homogenization ([23]).

5.2 Notation

- The maximum of two numbers a, b, will be denoted by $a \vee b$.

- For two positive integers M, N, we denote by $\mathcal{M}_{M\times N}$ the space of matrices of order $M \times N$. We also define by $\mathcal{M}_N^s$ the space of symmetric matrices of order $N \times N$.

- The space of linear applications (tensors) from $\mathcal{M}_{M\times N}$ into itself is denoted by $\mathcal{T}_{M\times N}$ and the space of linear applications from $\mathcal{M}_N^s$ into itself by $\mathcal{T}_{N,s}$.

- The scalar product of two vectors $a, b \in \mathbb{R}^M$ is denoted by $a \cdot b$ and the scalar product of two matrices $A, B \in \mathcal{M}_{M\times N}$ by $A : B$.

- We denote by $\mathcal{V}_M$ the set of linear subspaces of $\mathbb{R}^M$.

- For a function u, we denote by $e(u)$ the symmetric part of the derivative of u.

- Let us denote by Ω a Lipschitz bounded open subset of $\mathbb{R}^N$ and by $\widetilde{\Omega}$ another bounded open subset of $\mathbb{R}^N$ such that $\overline{\Omega} \subset \widetilde{\Omega}$. We recall the existence of a continuous linear extension operator $P : H^1(\Omega)^M \to H_0^1(\widetilde{\Omega})^M$.

- For $\rho > 0$ we denote

$$\Omega^{\rho^-} = \{x \in \Omega : d(x, \partial\Omega) < \rho\}, \quad \Omega^{\rho^+} = \left\{x \in \mathbb{R}^N : d(x, \overline{\Omega}) < \rho\right\}.$$

- We denote by $M(\overline{\Omega})$ the space of Radon measures in $\overline{\Omega}$.

  For every Borel set $E \subset \overline{\Omega}$ we define the capacity of E (with respect to $\widetilde{\Omega}$) by

$$cap(E) = \inf\left\{\int_{\widetilde{\Omega}} |\nabla u|^2\, dx : \quad u \in H_0^1(\widetilde{\Omega}), \quad u \geq 1 \text{ on a neighborhood of } E\right\}. \quad (5.7)$$

  Remark that although this definition of capacity depends on $\widetilde{\Omega}$, the fact that a subset of $\overline{\Omega}$ has capacity zero is independent of the choice of $\widetilde{\Omega}$.

- We say that a property holds *quasi everywhere* (we write q.e.) in a set $E \subset \overline{\Omega}$ if it holds in $E \setminus N$ with $\text{cap}(N) = 0$.

- A function $u : \overline{\Omega} \to \mathbb{R}$ is said to be quasi continuous if for every $\varepsilon > 0$ there exists a set $A \subset \overline{\Omega}$ with $cap(A) < \varepsilon$ such that the restriction of u to $\overline{\Omega} \setminus A$ is continuous. We recall ([22]) that every $u \in H^1(\Omega)$ admits a quasi continuous representative in $\overline{\Omega}$ which is unique up to a sets of null capacity. Throughout this paper we shall use such a quasi continuous representative to individuate an element of $H^1(\Omega)$.

- A subset E of $\overline{\Omega}$ is said to be quasi closed if for every $\varepsilon > 0$, there exists a closed set $F_\varepsilon \subset E$, such that $cap(E \setminus F_\varepsilon) < \varepsilon$. A subset E of $\overline{\Omega}$ is said to be quasi open if $\overline{\Omega} \setminus E$ is quasi closed.

5.3 Homogenization Results

Along this section we consider a Lipschitz bounded open set $\Omega \subset \mathbb{R}^N$ and a sequence of Lipschitz open sets $\Omega_n \subset \mathbb{R}^N$ such that for every $\rho > 0$ there exists $n_0 \in \mathbb{N}$ satisfying

$$\Omega^{\rho^-} \subset \Omega_n \subset \Omega^{\rho^+}, \quad \forall\, n \geq n_0, \tag{5.8}$$

Given a sequence of measures $\mu_n \in M(\overline{\Omega}_n)$ which vanish on the sets of zero capacity, a sequence of μ_n-measurable functions $R_n : \overline{\Omega}_n \to \mathcal{M}_{M\times N}$, such that there exists $\beta > 0$ satisfying

$$R_n\xi\cdot\xi \geq 0,\ |R_n\xi\cdot\eta| \leq \beta\,(R_n\xi\cdot\xi)^{\frac{1}{2}}\,(R_n\eta\cdot\eta)^{\frac{1}{2}}, \quad \forall\,\xi,\eta\in\mathbb{R}^M,\ \mu_n\text{-a.e. in } \overline{\Omega}_n \tag{5.9}$$

and a sequence of applications $V_n : \overline{\Omega}_n \to \mathcal{V}_M$, we denote by $\mathcal{D}_n$ the space

$$\mathcal{D}_n = \left\{ v \in H^1(\Omega_n)^M :\ v \in V_n \text{ q.e. in } \overline{\Omega}_n,\ \int_{\overline{\Omega}_n} R_n v\cdot v\, d\mu_n < +\infty \right\}. \tag{5.10}$$

We assume there exists ρ_n converging to zero such

$$\|v\|^2_{L^2(\Omega_n\setminus\Omega)} \leq \rho_n \left(\|v\|^2_{H^1(\Omega_n)^M} + \int_{\overline{\Omega}_n} R_n v\cdot v\, d\mu_n \right), \quad \forall\, v\in\mathcal{D}_n. \tag{5.11}$$

We also consider a matrix function $A \in L^\infty(\widetilde{\Omega};\mathcal{T}^{M\times N})$ such that there exists $\alpha > 0$ (which does not depend on n) satisfying

$$\alpha\|v\|^2_{H^1(\Omega_n)^M} \leq \int_{\Omega_n} ADv : Dv dx + \int_{\overline{\Omega}_n} R_n v\cdot v d\mu_n, \quad \forall\, v\in D_n. \tag{5.12}$$

In these conditions, our main result is the following homogenization theorem

Theorem 5.1 *There exist a subsequence of n, still denoted by n, a measure $\mu \in M(\overline{\Omega})$ which vanishes on the sets of null capacity, a μ-measurable function $R : \overline{\Omega} \to \mathcal{M}_{M\times M}$, such that*

$$R(x)\xi\cdot\xi \geq 0, \quad \forall\,\xi\in\mathbb{R}^M,\ \mu\text{-a.e. } x\in\overline{\Omega}, \tag{5.13}$$

$$R(x)\xi\cdot\eta \leq \gamma\,(R(x)\xi\cdot\xi)^{\frac{1}{2}}\,(R(x)\eta\cdot\eta)^{\frac{1}{2}}, \quad \forall\,\xi,\eta\in\mathbb{R}^M,\ \mu\text{-a.e. } x\in\overline{\Omega}, \tag{5.14}$$

for some constant $\gamma > 0$, and an application $V : \overline{\Omega} \to \mathcal{V}_M$ such that denoting by $\mathcal{D}$ the space

$$\mathcal{D} = \left\{ v\in H^1(\Omega)^M :\ v\in V \text{ q.e. in } \overline{\Omega},\ \int_{\overline{\Omega}} Rv\cdot v\, d\mu < +\infty \right\}, \tag{5.15}$$

we have that for every $f_n \in L^2(\Omega_n)^M$, $G_n \in L^2(\Omega_n)^{M\times N}$, $f \in L^2(\Omega)^M$ *and* $G \in L^2(\Omega)^{M\times N}$, *which satisfy*

$$\|f_n\|_{L^2(\Omega_n)^N} \leq C, \tag{5.16}$$

$$\lim_{\rho\to 0}\limsup_{n\to\infty}\int_{\Omega_n\setminus\Omega^{\rho^-}} |G_n|^2 dx = 0, \tag{5.17}$$

$$f_n \rightharpoonup f \ \text{in}\ L^2(\Omega^{\rho^-})^M, \quad G_n \to G \ \text{in}\ L^2(\Omega^{\rho^-})^{M\times N}, \quad \forall\, \rho > 0, \tag{5.18}$$

the unique solution u_n *of the variational problem*

$$\begin{cases} u_n \in \mathcal{D}_n \\ \displaystyle\int_{\Omega_n} ADu_n : Dv\, dx + \int_{\Omega_n} R_n u_n \cdot v\, d\mu_n = \int_{\Omega_n} f \cdot v\, dx + \int_{\Omega_n} G : Dv\, dx, \quad \forall\, v \in \mathcal{D}_n, \end{cases} \tag{5.19}$$

converges weakly in $H^1(\Omega^{\rho^-})$, *for every* $\rho > 0$, *to the unique solution* u *of*

$$\begin{cases} u \in \mathcal{D} \\ \displaystyle\int_{\Omega} ADu : Dv\, dx + \int_{\Omega} Ru \cdot v\, d\mu = \int_{\Omega} f \cdot v\, dx + \int_{\Omega} G : Dv\, dx, \quad \forall\, v \in \mathcal{D}. \end{cases} \tag{5.20}$$

In order to give an example, where the assumptions of Theorem 5.1 are satisfied, we consider the linear elasticity system on rugous domains.

Theorem 5.2 *For a Lipschitz bounded domain* $\omega \subset \mathbb{R}^{N-1}$ *and a bounded sequence* ψ_n *in* $W^{1,\infty}(\omega)$, *which converges uniformly to zero, we denote by* Ω_n *the open sets*

$$\Omega_n = \left\{x = (x', x_N) \in \mathbb{R}^{N-1} \times \mathbb{R} : \ x' \in \omega, \quad 0 < x_N < 1 + \psi(x')\right\} \tag{5.21}$$

and by Ω *the open set* $\Omega = \omega \times (0, 1)$. *We consider a tensor* $B \in L^\infty(\widetilde{\Omega}; \mathcal{T}_{N,s})$ *(*$\widetilde{\Omega}$ *open,* $\overline{\Omega} \subset \widetilde{\Omega}$*) such that there exists* $\alpha > 0$ *satisfying*

$$B(x)\xi : \xi \geq \alpha|\xi|^2, \quad \forall\, \xi \in \mathcal{M}_{N,s}, \ \text{a.e. } x \in \widetilde{\Omega}. \tag{5.22}$$

We consider a sequence of applications $V_n : \overline{\Omega}_n \to \mathcal{V}_N$ *such that* $V_n(x', 0) = \{0\}$ *for every* $x' \in \omega$. *Then, there exists a subsequence of* n, *still denoted by* n, *a measure* $\mu \in M(\overline{\Omega})$ *which vanishes on the sets of null capacity, a* μ*-measurable function* $R : \overline{\Omega} \to \mathbb{R}^M$, *such that*

$$R(x)\xi : \xi \geq 0, \quad \forall\, \xi \in \mathbb{R}^M, \ \mu\text{-a.e. } x \in \overline{\Omega}, \tag{5.23}$$

$$R(x)\xi : \eta \leq \gamma\,(R(x)\xi : \xi)^{\frac{1}{2}}\,(R(x)\eta : \eta)^{\frac{1}{2}}, \quad \forall\, \xi, \eta \in \mathbb{R}^M, \ \mu\text{-a.e. } x \in \overline{\Omega}, \tag{5.24}$$

for some constant $\gamma > 0$, and an application $V : \overline{\Omega} \to \mathcal{V}_N$, with $V(x', 0) = \{0\}$ for every $x' \in \omega$ such that defining $\mathcal{D}$ by (5.15), we have that for every $f_n \in L^2(\Omega_n)^M$, $G_n \in L^2(\Omega_n)^{M\times N}$, $f \in L^2(\Omega)^M$ and $G \in L^2(\Omega)^{M\times N}$, which satisfy (5.16), (5.17) and (5.18), the unique solution u_n of the variational problem

$$\begin{cases} u_n \in H^1(\Omega_n), \quad u_n \in V_n \ q.e.\ in\ \Omega_n \\ \displaystyle\int_{\Omega_n} Be(u_n) : e(v)\,dx = \int_{\Omega_n} f_n \cdot v\,dx + \int_{\Omega_n} G_n : e(v)\,dx \\ \forall\, v \in H^1(\Omega_n), \quad v \in V_n(x)\ q.e.\ in\ \Omega, \end{cases} \tag{5.25}$$

converges in $H^1(\Omega^{\rho^-})$, for every $\rho > 0$ to the unique solution u of the variational problem

$$\begin{cases} u \in D \\ \displaystyle\int_{\Omega} Be(u) : e(v)\,dx + \int_{\Omega} Ru \cdot v\,d\mu = \int_{\Omega} f \cdot v\,dx + \int_{\Omega} G : e(v)\,dx, \quad \forall\, v \in D. \end{cases} \tag{5.26}$$

Remark 5.3 *Taking $V_n = \mathbb{R}^N$ in Ω_n, problem (5.25) can be written as*

$$\begin{cases} -\operatorname{div}(Be(u_n) - G) = f \quad in\ \Omega_n \\ u_n = 0 \quad on\ \omega \times \{0\} \\ u_n \in V_n,\ (Be(u_n) - G)\cdot \nu \in V_n^{\perp}\ on\ \partial\Omega_n \setminus (\omega \times \{0\}), \end{cases}$$

This permits to study the behavior of a elasticity system in a rugous domain for several types of boundary conditions. As interesting cases we highlight for example

1. *V_n taking only the values $\{0\}$ or $\mathbb{R}^N$, this corresponds to assume that the elastic body is fixed in a variable portion of $\partial\Omega_n \setminus (\omega \times \{0\})$ and it is free on the rest.*
2. *$V_n(x)$ is equals to the tangent space to $\partial\Omega_n$ in the point x. This corresponds to a case where the elastic body is surrounded by other one which is very rigid and then it cannot penetrate it. Therefore, uniquely tangential deformations are possible on the boundary.*

5.4 Proof of the results.

The proof of Theorem 5.1 is based in the next abstract result. As we said in the introduction it is an adaptation to the H-covergence context of a result which appears in [16].

Theorem 5.4 *We consider a linear subspace $D \subset H^1(\Omega)^M \cap L^\infty(\Omega)^M$ and a non-negative bilinear form $\nu : D \times D \to M(\overline{\Omega})$ such that* *i) For every $\varphi \in C^1(\overline{\Omega})$ and every $u, v \in D$, the functions φu, φv belong to D and*

$$\nu(\varphi u, v) = \nu(u, \varphi v) = \varphi \nu(u, v) \quad \text{in } \overline{\Omega}. \tag{5.27}$$

ii) There exists $\beta > 0$ such that for every $u, v \in D$,

$$\|\nu(u, v)\|_{M(\overline{\Omega})} \leq \beta \|\nu(u, u)\|^{1/2}_{M(\overline{\Omega})} \|\nu(v, v)\|^{1/2}_{M(\overline{\Omega})}. \tag{5.28}$$

iii) There exists $\gamma \geq 1$ such that for every $u \in D$ and every sequence $u_n \in D$ converging weakly to u in $H^1(\Omega)^M$, we have

$$\|\nu(u, u)\|_{M(\overline{\Omega})} \leq \gamma \liminf_{n \to \infty} \left(\|\nu(u_n, u_n)\|_{M(\overline{\Omega})} + \|u_n\|^2_{H^1(\Omega)^M} \right). \tag{5.29}$$

Then, there exist $V : \overline{\Omega} \to \mathcal{V}_N$, $\mu \in M(\overline{\Omega})$, which vanishes on sets of capacity zero, and $R : \overline{\Omega} \to \mathcal{M}_{M \times N}$, μ-measurable, with

$$R\xi \cdot \xi \geq 0, \quad \forall \xi \in \mathbb{R}^M, \ \mu\text{-a.e. in } \overline{\Omega} \tag{5.30}$$

$$R\xi \cdot \eta \leq \beta \left(R\xi \cdot \xi \right)^{\frac{1}{2}} \left(R\eta \cdot \eta \right)^{\frac{1}{2}}. \quad \forall \xi, \eta \in \mathbb{R}^M, \ \mu\text{-a.e. in } \overline{\Omega}, \tag{5.31}$$

such that denoting by $\overline{D}$ the space

$$\overline{D} = \left\{ u \in H^1(\Omega)^M : \ u \in V \ \text{q.e. in } \overline{\Omega}, \ \int_{\overline{\Omega}} Ru \cdot u \, d\mu < +\infty \right\}, \tag{5.32}$$

we have

a) The space $\overline{D}$ is a Hilbert space endowed with the scalar product

$$(u, v)_{\overline{D}} = (u, v)_{H^1(\Omega)^M} + \frac{1}{2} \int_{\overline{\Omega}} (R + R^t) u \cdot v \, d\mu, \quad \forall u, v \in \overline{D}. \tag{5.33}$$

b) The space D is a dense subspace of $\overline{D}$ and

$$\nu(u, v) = Ru \cdot v \, d\mu \quad \forall u, v \in D. \tag{5.34}$$

Proof. The proof will be divided in two parts. In the first one we show the existence of $V : \overline{\Omega} \to \mathcal{V}_N$, $\mu \in M(\overline{\Omega})$, which vanishes on sets of zero capacity, and $R : \overline{\Omega} \to \mathcal{M}_{M \times N}$, μ-measurable, such that D is contained in the space

$$\left\{ u \in H^1(\Omega)^M : \ u \in V \ \text{q.e. in } \overline{\Omega}, \ \int_{\overline{\Omega}} Ru \cdot u \, d\mu < +\infty \right\}$$

and (5.34) holds. In the second part we will show that D is dense in the space defined above. Both parts will be divided in several steps.

FIRST PART.

Step 1. Let us take $\overline{D}$ as the completion of the space D endowed with the scalar product

$$(u,v)_D = (u,v)_{H^1(\Omega)^M} + \frac{1}{2}\int_{\overline{\Omega}} (d\nu(u,v) + d\nu(v,u)). \tag{5.35}$$

Taking into account that a Cauchy sequence in D is also a Cauchy sequence in $H^1(\Omega)^M$ and the semicontinuity property (5.29) we will prove that this completion can be carried out by defining $\overline{D}$ as the space of functions $u \in H^1(\Omega)^M$ such that there exists a Cauchy sequence u_n in D which converges to u in $H^1(\Omega)^M$ and extending ν to $\overline{\nu} : \overline{D} \times \overline{D} \to M(\overline{\Omega})$ by taking

$$\overline{\nu}(u,v) = \lim_{n\to\infty} \nu(u_n, v_n) \text{ in } M(\overline{\Omega}), \tag{5.36}$$

where $u_n, v_n \in D$ are Cauchy sequences in D which converge in $H^1(\Omega)^M$ to u and v respectively.

Let us check that the definition of $\overline{\nu}$ is correct, i.e. that the limit in the right-hand side of (5.36) exists and it does not depend on the sequences u_n, v_n chosen. In particular, we will deduce that $\overline{\nu}$ is an extension of ν.

The existence of the limit in (5.36) easily follows by using the inequality

$$\begin{aligned}
&\|\nu(u_n,v_n) - \nu(u_m,v_m)\|_{M(\overline{\Omega})} \le \|\nu(u_n - u_m, v_n)\|_{M(\overline{\Omega})} + \|\nu(u_m, v_n - v_m)\|_{M(\overline{\Omega})} \\
&\le \beta\Big(\|\nu(u_n - u_m, u_n - u_m)\|^{1/2}_{M(\overline{\Omega})} \|\nu(v_n,v_n)\|^{1/2}_{M(\overline{\Omega})} \\
&\qquad\qquad + \|\nu(u_m,u_m)\|^{1/2}_{M(\overline{\Omega})} \|\nu(v_n - v_m, v_n - v_m)\|^{1/2}_{M(\overline{\Omega})}\Big).
\end{aligned}$$

which taking into account that u_n, v_n are bounded in D and so $\nu(u_n,u_n)$, $\nu(v_n,v_n)$ are bounded in $M(\overline{\Omega})$, implies that $\nu(u_n,v_n)$ is a Cauchy sequence in $M(\overline{\Omega})$.

To prove that the definition of $\overline{\nu}$ does not depend on the sequences u_n, v_n considered, we remark that if u_n, $\widetilde{u}_n$ are two Cauchy sequences in D which converge in $H^1(\Omega)^M$ to the same function u, then (5.29), $u_n - \widetilde{u}_n$ converging to zero in $H^1(\Omega)^M$ and (5.28) give

$$\begin{aligned}
&\|\nu(u_m - \widetilde{u}_m, u_m - \widetilde{u}_m\|_{M(\overline{\Omega})} \\
&\le \gamma \liminf_{n\to\infty} \|\nu(u_n - \widetilde{u}_n - (u_m - \widetilde{u}_m), u_n - \widetilde{u}_n - (u_m - \widetilde{u}_m))\|_{M(\overline{\Omega})} + \gamma\|u_m - \widetilde{u}_m\|^2_{H^1(\Omega)^M} \\
&\le C \liminf_{n\to\infty} \Big(\|\nu(u_n - u_m, u_n - u_m)\|_{M(\overline{\Omega})} + \|\nu(\widetilde{u}_n - \widetilde{u}_m, \widetilde{u}_n - \widetilde{u}_m)\|_{M(\overline{\Omega})}\Big) + \gamma\|u_m - \widetilde{u}_m\|^2_{H^1(\Omega)^M}
\end{aligned}$$

Therefore, taking the limit in m, we get

$$\nu(u_m - \widetilde{u}_m, u_m - \widetilde{u}_m) \to 0 \text{ in } M(\overline{\Omega}). \tag{5.37}$$

Using (5.37) for $u_m - \widetilde{u}_m$ and $v_m - \widetilde{v}_m$, and the inequality

$$\|\nu(u_n, v_n) - \nu(\widetilde{u}_n, \widetilde{v}_n)\|_{M(\overline{\Omega})} \leq \|\nu(u_n - \widetilde{u}_n, v_n)\| + \|\nu(\widetilde{u}_n, v_n - \widetilde{v}_n)\|$$
$$\leq \beta \left(\|\nu(u_n - \widetilde{u}_n, u_n - \widetilde{u}_n)\|^{1/2}_{M(\overline{\Omega})} \|\nu(v_n, v_n)\|^{1/2}_{M(\overline{\Omega})} + \|\nu(\widetilde{u}_n, \widetilde{u}_n)\|^{1/2}_{M(\overline{\Omega})} \|\nu(v_n - \widetilde{v}_n, v_n - \widetilde{v}_n)\|^{1/2}_{M(\overline{\Omega})} \right)$$

we deduce that definition (5.36) of $\overline{\nu}$ does not depend on the sequences u_n, v_n chosen.

Step 2. Let us prove that $\overline{D}$ is a Hilbert space endowed with the scalar product

$$(u, v)_{\overline{D}} = (u, v)_{H^1_0(\Omega)^M} + \frac{1}{2} \left(\int_{\overline{\Omega}} d\overline{\nu}(v, u) + \int_{\overline{\Omega}} d\overline{\nu}(u, v) \right), \quad \forall\, u, v \in \overline{D}$$

and that D is dense in $\overline{D}$. This will prove that $\overline{D}$ is the completion of D as we mentioned in Step 1.

To prove the density we use that, by definition of $\overline{D}$, for every $u \in \overline{D}$ there exists a Cauchy sequence u_n in D which converges to u in $H^1(\overline{\Omega})^M$. Since definition (5.36) of $\overline{\nu}$ implies

$$\overline{\nu}(u_n - u, u_n - u) = \lim_{m \to \infty} \nu(u_n - u_m, u_n - u_m) \text{ in } M(\overline{\Omega}),$$

we then have that u_n converges to u in $\overline{D}$.

In order to show that $\overline{D}$ is complete, we consider a Cauchy sequence u_n in $\overline{D}$. Since this sequence is also a Cauchy sequence in $H^1(\Omega)^M$, there exists $u \in H^1(\Omega)^M$ such that u_n converges to u in $H^1(\Omega)^M$. On the other hand, thanks to the density of D in $\overline{D}$, there exits a sequence $\widetilde{u}_n \in \overline{D}$ such that $u_n - \widetilde{u}_n$ tends to zero in $\overline{D}$. Clearly, $\widetilde{u}_n$ is a Cauchy sequence in D which converges to u in $H^1(\Omega)^M$ and thus $u \in \overline{D}$. Using now that by definition of $\overline{\nu}$

$$\limsup_{n \to \infty} \|u_n - u\|_{\overline{D}} = \limsup_{n \to \infty} \|\widetilde{u}_n - u\|_{\overline{D}} = \limsup_{n \to \infty} \lim_{m \to \infty} \|\widetilde{u}_n - \widetilde{u}_m\|_D = 0,$$

we deduce that u_n converges to u in $\overline{D}$.

Step 3. Let us prove that in (5.29), we can take $\gamma = 1$. We consider a sequence $u_n \in D$ which converges weakly to a function $u \in D$ in $H^1(\Omega)^M$ and it is such that $\nu(u_n, u_n)$ is bounded (in other case (5.29) clearly holds with $\gamma = 1$). Then, u_n is bounded in the Hilbert space $\overline{D}$ and so there exists $\hat{u} \in \overline{D}$ such that, for a subsequence still denoted by n, u_n converges weakly in $\overline{D}$ to $\hat{u}$. Since u_n also converges to u in $H^1(\Omega)^M$, we get that $\hat{u} = u$. Now, the semicontinuity of the norm for the weak topology in $\overline{D}$ implies that (5.29) holds with $\gamma = 1$.

Step 4. Let us prove that for every $u, v \in D$, the measure $\nu(u, v)$ vanishes on the sets of null capacity.

By property (5.28), it is enough to consider the case $u = v$. Because $\nu(u,u)$ is a non-negative Radon measure, we have

$$\int_B d\nu(u,u) = \sup\left\{\int_K d\nu(u,u): \ K \subset B \text{ compact}\right\}, \quad \forall\, B \subset \overline{\Omega}, \text{ Borel}$$

and thus it is enough to show that every compact set K with null capacity satisfies

$$\int_K d\nu(u,u) = 0.$$

In this case, we know

$$cap(K) = \inf\left\{\int_{\widetilde{\Omega}} |\nabla\varphi|^2 dx: \qquad \varphi \in C_c^\infty(\widetilde{\Omega}), \quad \varphi = 1 \text{ in } K, \quad 0 \le \varphi \le 1 \text{ in } \widetilde{\Omega}\right\}.$$

So, if $cap(K) = 0$, there exists a sequence $\varphi_n \in C^\infty(\overline{\Omega})$ such that $0 \le \varphi_n \le 1$ in $\overline{\Omega}$, $\varphi_n = 1$ in K and φ_n converges to zero in $H^1(\overline{\Omega})$. From (5.29), with $\gamma = 1$, and (5.27), we have for every $k \in \mathbb{N}$

$$\begin{aligned}
\int_{\overline{\Omega}} \varphi_k^2 d\nu(u,u) &\le \liminf_{n\to\infty}\left(\int_{\overline{\Omega}} d\nu((1-\varphi_n)\varphi_k u, (1-\varphi_n)\varphi_k u) + \|(1-\varphi_n)\varphi_k u\|^2_{H^1(\Omega)^M}\right)\\
&= \liminf_{n\to\infty} \int_{\overline{\Omega}} (1-\varphi_n)^2 \varphi_k^2 d\nu(u,u) + \|\varphi_k u\|^2_{H^1(\Omega)^M}\\
&\le \liminf_{n\to\infty} \int_{\overline{\Omega}} (1-\varphi_n) \varphi_k^2 d\nu(u,u) + \|\varphi_k u\|^2_{H^1(\Omega)^M}\\
&= \int_{\overline{\Omega}} \varphi_k^2 d\nu(u,u) - \limsup_{n\to\infty} \int_{\overline{\Omega}} \varphi_n \varphi_k^2 d\nu(u,u) + \|\varphi_k u\|^2_{H^1(\Omega)^M}.
\end{aligned}$$

Thus, using that $\chi_K \le \varphi_n \varphi_k^2$, we get

$$\int_K d\nu(u,u) \le \limsup_{n\to\infty} \int_{\overline{\Omega}} \varphi_n \varphi_k^2 d\nu(u,u) \le \|\varphi_k u\|^2_{H^1(\Omega)^M},$$

which taking the limit in k proves the result.

Step 5. It is immediate to check that properties (5.27), (5.28) and (5.29) still hold true with D and ν replaced by $\overline{D}$ and $\overline{\nu}$ respectively. Moreover $\overline{\nu}(u,v)$ vanishes on the sets of capacity zero for every $u, v \in \overline{D}$. Let us prove that property (5.27) can be improved in the following way: For every $\varphi \in H^1(\Omega) \cap L^\infty(\Omega)$ and every $u, v \in \overline{D} \cap L^\infty(\Omega)^M$, the functions φu, φv belong to $\overline{D}$ and

$$\overline{\nu}(\varphi u, v) = \overline{\nu}(u, \varphi v) = \varphi\overline{\nu}(u,v). \tag{5.38}$$

For this purpose, we use that for every $\varphi \in H^1(\Omega)\cap L^\infty(\Omega)$, there exists $\varphi_n \in C^1(\overline{\Omega})$ which is bounded in $L^\infty(\Omega)$ and converges to φ in $H^1(\Omega)$ and q.e. in $\overline{\Omega}$. Taking into account that, for every $u \in \overline{D}\cap L^\infty(\Omega)^M$, $\overline{\nu}(u,u)$ is a bounded measure which vanishes on sets of null capacity, we get that φ_n converges to φ in $L^2(\Omega, d\nu(u,u))$, which by (5.27) and $\varphi_n u$ converging to φu in $H^1(\Omega)^M$ implies that $\varphi_n u$ is a Cauchy sequence in D and therefore that φu belongs to $\overline{D}$.

Now, for $u, v \in D$, passing to the limit in $\nu(\varphi_n u, v) = \nu(u, \varphi_n v) = \varphi_n \overline{\nu}(u,v)$ we deduce (5.38).

Step 6. Let us prove that $\overline{\nu}(u,v)$ depends locally on the values of u, v in the sense that for every $u, v \in \overline{D} \cap L^\infty(\Omega)^M$ and every $w \in \overline{D}$, we have

$$\int_{\{u=v\}} d|\overline{\nu}(u-v,w)| = 0. \tag{5.39}$$

Thanks to (5.28) and $\overline{\nu}$ bilinear, it is enough to show that every function $u \in \overline{D} \cap L^\infty(\Omega)^M$ satisfies

$$\int_{\{u=0\}} d\overline{\nu}(u,u) = 0. \tag{5.40}$$

In order to prove (5.40), given $\varepsilon > 0$, we take $\varphi_\varepsilon \in H^1(\Omega) \cap L^\infty(\Omega)$ as $\varphi_\varepsilon = \varepsilon/(|u| \vee \varepsilon)$. Then, by Step 4 the sequence $u_\varepsilon = \varphi_\varepsilon u$ is in $\overline{D}$ (remark that u_ε is the projection of u on the ball of center zero and radius ε). Clearly it converges to zero in $L^\infty(\Omega)^M$ and using that for every $\delta > \varepsilon$ one has

$$|Du_\varepsilon| \leq |Du|\chi_{\{|u|<\varepsilon\}} + \frac{2\varepsilon}{|u|}|Du|\chi_{\{\varepsilon\leq|u|\}} \leq 2|Du|\chi_{\{|u|<\delta\}} + \frac{2\varepsilon}{\delta}|Du|\chi_{\{\delta\leq|u|\}},$$

we easily deduce (taking the limit in ε and then in δ) that Du_ε tends to zero in $L^2(\Omega)^{M\times N}$. Thus, u_ε tends to zero in $H^1(\Omega)^M$. On the other hand, by (5.38) we have that $\overline{\nu}(u_\varepsilon, u_\varepsilon) = \varphi_\varepsilon^2\overline{\nu}(u,u) \leq \overline{\nu}(u,u)$ in $\overline{\Omega}$, which proves that u_ε is bounded in $\overline{D}$. Using that $\overline{D}$ is reflexive and it is continuously embedded in $H^1(\Omega)^M$, we then deduce that u_ε converges weakly to zero in $\overline{D}$. By Mazur's theorem, we can then extract a sequence ψ_ε of convex combinations of the functions φ_ε such that $\psi_\varepsilon u$ converges strongly to u in $\overline{D}$. Since $\varphi_\varepsilon = 1$ q.e. in the set $\{u = 0\}$ and $\varphi_\varepsilon \geq 0$ in $\overline{\Omega}$, we get that also $\psi_\varepsilon = 1$ q.e. in the set $\{u = 0\}$ and $\psi_\varepsilon \geq 0$ in $\overline{\Omega}$, which implies

$$\int_{\{u=0\}} d\nu(u,u) \leq \int_{\overline{\Omega}} |\psi_\varepsilon|^2 d\nu(u,u) \leq \|\psi_\varepsilon u\|_{\overline{D}}^2 \to 0.$$

Step 7. Since $H^1(\Omega)^M$ is separable and $\overline{D}$ is continuously embedded in $H^1(\Omega)^M$, we get that $\overline{D}$ is separable. We denote by $\{z_k\} \subset D$ a countable dense subset of $\overline{D}$. For q.e. $x \in \overline{\Omega}$ we define

$$V(x) = span\{z_k(x)\}. \tag{5.41}$$

Clearly, we have

$$u(x) \in V(x) \ \text{ for q.e. } x \in \overline{\Omega},\ \forall\, u \in \overline{D}. \tag{5.42}$$

We also define

$$V_m = \{x \in \overline{\Omega}\ : \dim V(x) = m\}, \quad 1 \leq m \leq M, \tag{5.43}$$

and

$$V_{i_1,\ldots,i_m} = \{x \in V_m(x)\ :\ z_{i_1},\ldots,z_{i_m} \text{ are linearly independent}\}, \quad 1 \leq i_1 < \ldots < i_m. \tag{5.44}$$

Denoting

$$\Omega^* = \bigcup_{m=1}^{N} \bigcup_{\{1\leq i_1<\ldots<i_m\}} V_{i_1,\ldots,i_m}$$

we have that

$$u = 0 \ \text{ q.e. in } \overline{\Omega}\setminus\Omega^*, \quad \forall\, u \in \overline{D}. \tag{5.45}$$

Fixed $m \in \mathbb{N}$ and $i_1,\ldots,i_m$ with $1 \leq i_1 < \ldots < i_m$, let us obtain an integral expression of $\overline{\nu}$ in $V_{i_1,\ldots,i_m}$.

Let u be in $\overline{D}$. By (5.42), we have

$$u(x) = \sum_{s=1}^{m} \varphi_s(x) z_{i_s}(x), \quad \text{for q.e. } x \in V_{i_1,\ldots,i_m}$$

where the functions $\varphi_1,\ldots,\varphi_m : V_{i_1,\ldots,i_m} \to \mathbb{R}$ are given by

$$\begin{pmatrix} \varphi_1(x) \\ \vdots \\ \varphi_m(x) \end{pmatrix} = Z^{-1} \begin{pmatrix} uz_{i_1} \\ \vdots \\ uz_{i_m} \end{pmatrix}, \tag{5.46}$$

with Z the matrix of entries $Z_{sr} = z_{i_s}\cdot z_{i_s}$. Since the functions $z_{i_1},\ldots,z_{i_m}$ are in the algebra $H^1(\Omega)^M \cap L^\infty(\Omega)^M$, we have that the determinant of any squared submatrix of Z is in $H^1(\Omega)\cap L^\infty(\Omega)$. Thus, denoting by $d = \det Z \in H^1(\Omega)\cap L^\infty(\Omega)$ we deduce that $d(x) > 0$ q.e. in $V_{i_1,\ldots,i_m}$ and

$$d(x)u(x) = \sum_{s=1}^{m} (w_s(x)\cdot u(x)) z_{i_s}(x), \quad \text{q.e. } x \in V_{i_1,\ldots,i_m} \tag{5.47}$$

where the functions w_s are in $H^1(\Omega)^M \cap L^\infty(\Omega)^M$ and they do not depend on u.

Taking now $u, v \in \overline{D} \cap L^\infty(\Omega)^M$ and using (5.39), $\overline{\nu}$ bilinear and (5.38), we get

$$d(x)^2 \overline{\nu}(u,v) = \sum_{s,r=1}^{m} (w_s(x) \cdot u(x))(w_r(x) \cdot v(x)) \nu(z_{i_s}, z_{i_r}) \ \text{ in } V_{i_1,\ldots,i_m}. \tag{5.48}$$

We define

$$\mu_{i_1,\ldots,i_m} = \sum_{s=1}^{m} \nu(z_{i_s}, z_{i_s}).$$

By (5.28), (5.38) and the derivation measures theorem, we deduce that there exist $h_{i_s i_r} \in L^\infty(\overline{\Omega}, d\mu_{i_1,\ldots,i_m})$, $1 \leq s, r \leq m$, such that

$$\nu(z_{i_s}, z_{i_r}) = h_{i_s i_r} \mu_{i_1,\ldots,i_m}.$$

Therefore, defining in $V_{i_1,\cdots,i_m}$

$$R_{i_1,\ldots,i_m} = \frac{1}{d^2} \sum_{s,r=1}^{m} w_s \otimes w_r \, h_{i_s} h_{i_r},$$

which is a non-negative matrix $\mu_{i_1,\cdots,i_m}$-measurable function we get

$$\overline{\nu}(u,v) = R_{i_1,\cdots,i_m} u \cdot v \, \mu_{i_1,\ldots,i_m} \ \text{ in } V_{i_1,\ldots,i_m}, \quad \forall\, u, v \in \overline{D} \cap L^\infty(\Omega)^M. \tag{5.49}$$

Step 8. In order to obtain a representation of $\overline{\nu}(u,v)$ in the whole of $\overline{\Omega}$, it is enough to consider Borel sets $W_{i_1,\ldots,i_m} \subset V_{i_1,\ldots,i_m}$ which are disjoint, and satisfy

$$\Omega^* = \bigcup_{m=1}^{N} \bigcup_{\{1 \leq i_1 < \ldots < i_m\}} W_{i_1,\ldots,i_m}.$$

Defining then the measure $\mu \in \mathcal{M}(\overline{\Omega})$

$$\mu(B) = \sum_{m=1}^{N} \sum_{\{1 \leq i_1 < \ldots < i_m\}} \mu_{i_1,\ldots,i_m} \left(B \cap W_{i_1,\ldots,i_m} \right), \quad \forall\, B \subset \overline{\Omega} \text{ Borel}$$

$$R = \sum_{m=1}^{N} \sum_{\{1 \leq i_1 < \ldots < i_m\}} R_{i_1,\ldots,i_m} \chi_{W_{i_1,\ldots,i_m}},$$

we conclude that

$$\overline{\nu}(u,v) = Ru \cdot v \, \mu, \quad \forall\, u, v \in \overline{D} \cap L^\infty(\Omega)^M. \tag{5.50}$$

If u, v are not in $L^\infty(\Omega)^M$, then, we consider $u_n, v_n \in D$ which respectively converge to u, v in $\overline{D}$. This means in particular that $\nu(u_n, v_n)$ is a Cauchy sequence in $M(\overline{\Omega})$ and then that $Ru_n \cdot v_n$ is a Cauchy sequence in $L^1_\mu(\overline{\Omega})$. On the other hand, since u_n, v_n converge strongly to u, v in $H^1(\Omega)^M$, we get that they converge to u, v in μ-measure and so, we deduce that $Ru \cdot v$ belongs to $L^1_\mu(\overline{\Omega})$ and

$$\overline{\nu}(u,v) = \lim_{n\to\infty} \overline{\nu}(u_n, v_n) = Ru \cdot v\, \mu, \quad \forall\, u, v \in \overline{D}. \tag{5.51}$$

Clearly, R is nonnegative and taking into account that (5.28) and (5.27) imply

$$\int_{\overline{\Omega}} Ru \cdot v\, \varphi\, d\mu \leq \left(\int_{\overline{\Omega}} Ru \cdot u\, \varphi\, d\mu\right)^{\frac{1}{2}} \left(\int_{\overline{\Omega}} Rv \cdot v\, \varphi\, d\mu\right)^{\frac{1}{2}},$$

for every $u, v \in \overline{D}$ and every $\varphi \in C^1(\overline{\Omega}, \varphi \geq 0$, we can show that R satisfies (5.31).

SECOND PART.

To prove that D is dense in

$$\left\{ u \in H^1(\Omega)^M :\ u \in V \text{ for q.e. in } \overline{\Omega},\ \int_{\overline{\Omega}} Ru \cdot u\, d\mu < +\infty \right\}, \tag{5.52}$$

it is enough to show that the space $\overline{D}$ defined in the first part agrees with (5.52); i.e. that every function $u \in H^1(\Omega)^M$ with $u(x) \in V(x)$ q.e. in $\overline{\Omega}$ and $Ru \cdot u \in L^1(\overline{\Omega}, d\mu)$ is in $\overline{D}$. By density, it easy to check that it is enough to consider the case where u also belongs to $L^\infty(\Omega)^M$.

Step 9. Let us start by proving the following result we will need later: Assume $w \in H^1(\Omega)^M \cap L^\infty(\Omega)^M$, $\psi \in H^1(\Omega) \cap L^\infty(\Omega)$, $\psi \geq 0$ in Ω such that

$$w = 0 \ \text{ q.e. in } \{\psi = 0\}, \quad \psi w \in \overline{D}, \quad \int_{\overline{\Omega}} Rw \cdot w\, d\mu < +\infty, \tag{5.53}$$

then w belongs to $\overline{D}$.

To prove this result, we take $\delta > 0$. Since for $\varepsilon > 0$, the sets

$$F_\varepsilon = \{x \in \overline{\Omega} :\ \psi(x) \leq \varepsilon,\ |w(x)| \geq \delta\},$$

are quasi-closed and they are increasing in ε, we get that

$$\lim_{\varepsilon\to 0} cap(F_\varepsilon) = cap\left(\bigcap_{\varepsilon>0} F_\varepsilon\right) = 0.$$

Therefore, there exists a sequence $\varphi_\varepsilon \in H_0^1(\widetilde{\Omega})$ such that $0 \le \varphi_\varepsilon \le 1$ in $\widetilde{\Omega}$, $\varphi_\varepsilon = 1$ in F_ε and φ_ε tends to zero in $H_0^1(\widetilde{\Omega})$. Using then that

$$(1-\varphi_\varepsilon)\frac{(|w|-\delta)^+}{(|w|\vee\delta)(\psi\vee\varepsilon)}\psi w = (1-\varphi_\varepsilon)\frac{(|w|-\delta)^+}{|w|}w,$$

were

$$(1-\varphi_\varepsilon)\frac{(|w|-\delta)^+}{|w|(\psi\vee\varepsilon)}$$

belongs to $H^1(\Omega)\cap L^\infty(\Omega)$ and ψw belongs to $\overline{D}\cap H_0^1(\widetilde{\Omega})$, we deduce by Step 4 that

$$(1-\varphi_\varepsilon)\frac{(|w|-\delta)^+}{|w|}w,$$

belongs to $\overline{D}$. Moreover, taking into account that $Rw\cdot w$ belongs to $L^1(\overline{\Omega}, d\mu)$ and that φ_ε tends to zero in $H^1(\widetilde{\Omega})$, we deduce that this sequence is bounded in $\overline{D}$ and converges to $(|w|-\delta)^+w/|w|$. This implies that $(|w|-\delta)^+w/|w|$ belongs to $\overline{D}$. Similarly, we can now that $(|w|-\delta)^+w/|w|$ is bounded in $\overline{D}$ and converges to w in $H^1(\Omega)$ when δ tends to zero, to finally prove that w belongs to $\overline{D}$.

Step 10. The ideas we use in the following are inspired in [17]. We consider the space

$$V_u = \{\psi \in H^1(\Omega)\ :\ \psi u \in \overline{D}\} \tag{5.54}$$

endowed with the norm

$$\|\psi\|_{V_u}^2 = \|\psi\|_{H^1(\Omega)}^2 + \int_\Omega \psi^2|Du|^2dx + \int_{\overline{\Omega}} \psi^2 Ru\cdot u\, d\mu. \tag{5.55}$$

It is very simple to check that V_u is a Hilbert space. Moreover, thanks to Step 4, for every $\psi \in V_u\cap L^\infty(\Omega)$ and every $\varphi\in H^1(\Omega)\cap L^\infty(\Omega)$, the function $\psi\varphi$ belongs to V_u.

Let us show that this implies the following property

$$\begin{gathered}\forall\,\psi\in V_u\cap L^\infty(\Omega),\ \forall\,\varphi\in H^1(\Omega)\cap L^\infty(\Omega) \text{ such that}\\ \exists\,M>0,\ \text{with } |\varphi|\le M|\psi| \text{ q.e. in } \overline{\Omega},\ \text{one has } \varphi\in V_u.\end{gathered} \tag{5.56}$$

In order to prove (5.56) we first assume that there exists $\varepsilon>0$ such that φ vanishes q.e. on $\{x\in\overline{\Omega}:\ |\psi(x)|<\varepsilon\}$, then, using that

$$\varphi = \frac{\varphi}{|\psi|\vee\varepsilon}\psi$$

we get the result. In the general case we take

$$\varphi_\varepsilon = \frac{(\psi-\varepsilon)^+}{|\psi|}\varphi$$

which by the above proved is in V_u and converges q.e. in Ω to φ. Using then that $\varphi_\varepsilon u$ is bounded in $\overline{D}$ we deduce (5.56).

Step 11. Using Riesz's Theorem, we define $\hat{\psi} \in V_u$ by

$$(\hat{\psi}, \varphi)_{V_u} = \int_\Omega \varphi\, dx, \qquad \forall\, \varphi \in V_u$$

or equivalently

$$\int_\Omega \nabla\hat{\psi}\cdot\nabla\varphi\, dx + \int_\Omega \left(|Du|^2+1\right)\hat{\psi}\varphi\, dx + \int_{\overline{\Omega}} Ru\cdot u\hat{\psi}\varphi\, d\mu = \int_\Omega \varphi\, dx, \quad \forall\varphi\in V_u. \tag{5.57}$$

By (5.56) the functions $\hat{\psi}^-$ and $(\hat{\psi}-1)^+$ belong to V_u. Taking them as test function in (5.57) we show (as in the classical proof of the weak maximum principle) that $\hat{\psi}$ satisfies

$$0 \le \hat{\psi} \le 1 \ \text{ q.e. in } \overline{\Omega}. \tag{5.58}$$

Moreover, let us show that $\hat{\psi}$ satisfies the following property:

$$\text{If } \psi \in V_u, \text{ then } \psi = 0 \text{ q.e. in } \{x\in\overline{\Omega}:\ \hat{\psi}(x)=0\}. \tag{5.59}$$

For this purpose we use that the space W of functions $\psi \in V_u$ such that there exists $f \in L^\infty(\Omega)$ with

$$(\psi,\varphi)_{V_u} = \int_\Omega f\varphi\, dx, \quad \forall\varphi\in V_u \tag{5.60}$$

is dense in V_u. This assertion is immediate by observing that if a function φ is such that $(\psi,\varphi)_{V_u}=0$ for every $\psi\in W$, then φ is the zero function.

Moreover, if ψ belongs to W, then the classical proof of the maximum principle shows that

$$-\|f\|_{L^\infty(\Omega)}\hat{\psi} \le \psi \le \|f\|_{L^\infty(\Omega)}\hat{\psi} \ \text{ q.e. in } \overline{\Omega}.$$

Therefore, (5.59) holds for every $\psi\in W$ and then, by density (use that the convergence in V_u implies convergence q.e. for a subsequence) for every ψ in V_u.

Step 12. Let us show that $\hat{\psi}$ is strictly positive q.e. in Ω^*. Once this has been proved, Step 9 with $w=u$, $\psi=\hat{\psi}$ will imply that u belongs to $\overline{D}$ and then the proof of Theorem 5.4 will be done.

By (5.45), it is enough to show that $\hat{\psi}$ is strictly positive q.e. in every set $V_{i_1,\cdots,i_m}$ with $1 \leq m \leq N$. We reason by contradiction.

Let us first assume $m = N$. Denoting as in Step 6, Z the matrix of entries $Z_{sr} = z_{is} \cdot z_{is}$, and by d the determinant of the non-negative matrix Z, we remark that since $m = N$, the set $V_{i_1,\cdots,i_m}$ agrees with the set where d is strictly positive and therefore (5.47) holds true not only for q.e. $x \in V_{i_1,\cdots,i_m}$ but for q.e. $x \in \overline{\Omega}$. Since the second member of (5.47) is and element of $\overline{D} \cap L^\infty(\Omega)^M$, this implies that $d\,u$ belongs to $\overline{D}$, and therefore that d belongs to V_u which by (5.59) will prove that $\hat{\psi} > 0$ q.e. in $V_{i_1,\cdots,i_m}$.

We now consider $m < N$ and we assume by induction hypothesis that $\hat{\psi} > 0$ q.e. in

$$\bigcup_{j=m+1}^{N} \bigcup_{\{1 \leq i_1,\cdots \leq i_j\}} V_{i_i,\cdots,i_j} = \big\{x \in \overline{\Omega} : \ \mathrm{dim}V(x) > m\big\}.$$

For $1 \leq i_1, \cdots \leq i_m$, we define $u_{i_1,\cdots,i_m}$ as the right-hand side of (5.59) (which agrees with the orthogonal projection of du on the space generated by $z_{i_1}, \cdots, z_{i_m}$) and $w = du - u_{i_1,\cdots,i_m}$. By the assumptions on u, the function w is in $H^1(\Omega)^M \cap L^\infty(\Omega)^M$ and satisfies that $Rw \cdot w$ belongs to $L^1_\mu(\Omega)$. Since $d\,\hat{\psi}$ belongs to V_u, we also have that $\hat{\psi}w$ is in $\overline{D}$. Moreover on the set where $\hat{\psi}$ vanish, we have

- If $d = 0$ then $w = 0$.

- If $d \neq 0$, then $z_{i_1}, \cdots, z_{i_m}$ are linearly independent, and by the induction assumption that $\mathrm{dim}V(x) \leq m$. Therefore $\{\hat{\psi} = 0\} \cap \{\hat{d} > 0\}$ is contained in $V_{i_i,\cdots,i_m}$ and so, by (5.47) we also have $w = 0$.

We can then apply Step 9 with $\psi = \hat{\psi}$ to deduce that w and then that du is in $\overline{D}$, which by (5.59) shows that $\hat{\psi}$ is strictly positive q.e. on $\{d > 0\}$ and in particular in $V_{i_i,\cdots,i_m}$. □

In Theorem 5.4 we have considered a set D contained in $L^\infty(\Omega)^M$, but since in (5.19) we are working with a systems of equations and not with a simple equation, it is well known that even for f and G very smooth, the solutions of problem (5.19) are not in general in $L^\infty(\Omega)^M$. To overcome this difficulty, we will use the following Lemma which is based on the ideas used in [20] (see also [10])

Proposition 5.5 *We consider a sequence of Lipschitz bounded open sets $\Omega_n \subset \mathbb{R}^N$ such that for every $\rho > 0$ there exists $n_0 \in \mathbb{N}$ satisfying (5.8), a sequence of measures $\mu_n \in M(\overline{\Omega}_n)$ which vanish on the sets of $\overline{\Omega}_m$ of null capacity, a sequence of μ-measurable functions $R_n : \overline{\Omega}_n \to \mathcal{M}_{M\times N}$, satisfying (5.9), a sequence of applications $V_n : \overline{\Omega}_n \to \mathcal{V}_N$, and a matrix function $A \in L^\infty(\tilde{\Omega}; \mathcal{T}_{M\times N})$ which satisfies (5.12) for some $\alpha > 0$. Defining $\mathcal{D}_n$ by (5.10), we assume there exist $u_n \in \mathcal{D}_n$, $u \in H^1(\Omega)^M$, $S > 0$, with*

$$\limsup_{n\to\infty} \left(\|u_n\|^2_{H^1(\Omega_n)^M} + \int_{\overline{\Omega}_n} R_n u_n \cdot u_n \, d\mu_n \right) \leq S, \qquad u_n \rightharpoonup u \ \ \textit{in } H^1(\Omega^{\rho^-})^M, \quad \forall \rho > 0, \tag{5.61}$$

and $f\in L^2(\Omega)^M$, $G\in L^2(\Omega)^{M\times N}$, such that for every $v_n\in \mathcal{D}_n$, $v\in H^1(\Omega)^M$, with

$$\limsup_{n\to\infty}\left(\|v_n\|^2_{H^1(\Omega_n)^M}+\int_{\overline{\Omega}_n}R_nv_n\cdot v_n\,d\mu_n\right)<\infty,\qquad v_n\rightharpoonup v\ \text{ in } H^1(\Omega^{\rho^-})^M,\quad \forall\,\rho>0, \tag{5.62}$$

we have

$$\lim_{n\to\infty}\left(\int_{\Omega_n}ADu_n:Dv_n\,dx+\int_{\overline{\Omega}_n}R_nu_n\cdot v_n\,d\mu_n\right)=\int_\Omega f\cdot v\,dx+\int_\Omega G:Dv\,dx. \tag{5.63}$$

Then, for every nonnegative Λ and every positive integer k, we have

$$\begin{aligned}&\limsup_{n\to\infty}\left(\int_{\{|u_n|>2^k\Lambda\}}|Du_n|^2dx+\int_{\{|u_n|>2^k\Lambda\}}ADu_n:Du_ndx+\int_{\{|u_n|>2^k\Lambda\}}Ru_n:u_n\,d\mu_n\right)\\&\le C\left(\int_{\{|u|>\Lambda\}}|f||u|\,dx+\int_{\{|u|>\Lambda\}}|G||Du|\,dx+\frac{S}{k}\right).\end{aligned} \tag{5.64}$$

The constant C in (5.64) only depends on the applications R_n and A.

Proof. We take $\Lambda>0$ and $k\in\mathbb{N}$, thanks to (5.61) we have

$$\sum_{j=0}^{k-1}\int_{\{2^j\Lambda<|u_n|<2^{j+1}\Lambda\}}|Du_n|^2dx\le S+O_n, \tag{5.65}$$

where O_n tends to zero when n tends to infinity. Therefore, for every $n\in\mathbb{N}$ there exists $j_n\in\{0,\cdots,k-1\}$ such that

$$\int_{\{2^{j_n}\Lambda<|u_n|<2^{j_n+1}\Lambda\}}|Du_n|^2dx\le\frac{S+O_n}{k}. \tag{5.66}$$

Extracting a subsequence if necessary, we can also assume that j_n converges to some $j\in\{0,\cdots,k-1\}$.

We consider a function $\Phi\in C^\infty(\mathbb{R}^M)$ such that $\Phi(S)=0$ if $|s|<1$, $\Phi(s)=1$ if $|s|>2$, $0\le\Phi\le1$ in $\mathbb{R}^M$ and $|\nabla\Phi|\le2$ in $\mathbb{R}^M$. Using in (5.63) $v_n=\Phi(u_n/(2^{j_n}\Lambda))^2u_n$, $v=\Phi(u/(2^j\Lambda))^2u$, and taking into account

$$\left|D\left[\Phi(u_n/(2^{j_n}\Lambda))^2u_n\right]-\Phi(u_n/(2^{j_n}\Lambda))^2Du_n\right|\le4|Du_n|\chi_{\{2^{j_n}\Lambda<|u_n|<2^{j_n+1}\Lambda\}},\quad\text{a.e. in }\Omega_n, \tag{5.67}$$

and (5.66), we have

$$\begin{aligned}&\limsup_{n\to\infty}\left(\int_{\Omega_n}\Phi(\frac{u_n}{2^{j_n}\Lambda})^2ADu_n:Du_n\,dx+\int_{\overline{\Omega}_n}\Phi(\frac{u_n}{2^{j_n}\Lambda})^2Ru_n\cdot u_n\,d\mu_n\right)\\&\le\int_\Omega\Phi(\frac{u}{2^j\Lambda})^2f\cdot u\,dx+\int_\Omega G\cdot D\left[\Phi(\frac{u}{2^j\Lambda})^2u\right]+\frac{CS}{k},\end{aligned}$$

which, using that analogously to (5.68), we have

$$\left|D\left[\Phi(u_n/(2^{j_n}\Lambda))u_n\right]-\Phi(u_n/(2^{j_n}\Lambda))Du_n\right|\leq 4|Du_n|\chi_{\{2^{j_n}\Lambda<|u_n|<2^{j_n+1}\Lambda\}},\quad \text{a.e. in } \Omega_n, \tag{5.68}$$

implies

$$\begin{aligned}&\limsup_{n\to\infty}\left(\int_{\Omega_n}AD\left[\Phi(\frac{u_n}{2^{j_n}\Lambda})u_n\right]:D\left[\Phi(\frac{u_n}{2^{j_n}\Lambda})u_n\right]\,dx+\int_{\overline{\Omega}_n}R\left[\Phi(\frac{u}{2^{j}\Lambda})u\right]\cdot\left[\Phi(\frac{u}{2^{j}\Lambda})u\right]\,d\mu_n\right)\\&\leq\int_\Omega\Phi(\frac{u}{2^j\Lambda})^2f\cdot u\,dx+\int_\Omega G\cdot D\left[\Phi(\frac{u}{2^j\Lambda})^2u\right]+\frac{CS}{k}.\end{aligned} \tag{5.69}$$

Since

$$\int_\Omega\Phi(\frac{u}{2^j\Lambda})^2f\cdot u\,dx+\int_\Omega G\cdot D\left[\Phi(\frac{u}{2^j\Lambda})^2u\right]\leq\int_{\{|u|>\Lambda\}}|f||u|\,dx+C\int_{\{|u|>\Lambda\}}|G||Du|\,dx,$$

inequalities (5.68), (5.69) and (5.12) easily show (5.64). □

Taking in (5.64) k and then Λ tending to infinity,we immediately deduce from Proposition 5.5

Corollary 5.6 *In the conditions of Proposition 5.5, we have*

$$\lim_{m\to\infty}\limsup_{n\to\infty}\left(\int_{\{|u_n|>m\}}ADu_n:Du_ndx+\int_{\{|u_n|>m\}}Ru_n:u_n\,d\mu_n\right)=0. \tag{5.70}$$

Remark 5.7 *Assuming u in $L^\infty(\Omega)^M$, the same proof used above shows that Proposition 5.5 (and then Corollary 5.6) still holds true if we assume that f is only in $L^1(\Omega)^N$ and that (5.62) only holds when v is also in $L^\infty(\Omega)^M$.*

Proposition 5.8 *In the conditions of Proposition 5.5, we consider $\tau>0$ and a nonnegative sequence δ_n such that*

$$\delta_n\to 0,\quad \frac{1}{\delta_n^2}\int_{\Omega\cap\{|u_n-u|>1\}}|Du_n|^2dx\to 0, \tag{5.71}$$

and define $\varphi_n\in H^1(\Omega_n)\cap L^\infty(\Omega_n)$ by

$$\varphi_n=\frac{1}{1+\delta_n|u_n|+(\tau-\delta_n)|Pu|}, \tag{5.72}$$

with $Pu\in H_0^1(\widetilde{\Omega})$ and extension of u to $\widetilde{\Omega}$ Then, φ_nu_n satisfies

$$\lim_{m\to\infty}\limsup_{n\to\infty}\left(\int_{\{|u_n|>m\}}AD(\varphi_nu_n):D(\varphi_nu_n)dx+\int_{\{|u_n|>m\}}R(\varphi_nu_n):(\varphi_nu_n)\,d\mu_n\right)=0. \tag{5.73}$$

Moreover, for every $v_n \in H^1(\Omega_n)^M$, $v \in H^1(\Omega)^M \cap L^\infty(\Omega)^M$ which satisfy (5.62) and

$$\lim_{m\to\infty} \limsup_{n\to\infty} \left(\int_{\{|v_n|>m\}} |Dv_n|^2\,dx + \int_{\{|v_n|>m\}} Rv_n \cdot v_n\,d\mu_n \right) = 0 \tag{5.74}$$

we have

$$\begin{aligned} &\lim_{n\to\infty} \left(\int_{\Omega_n} AD(\varphi_n u_n) : Dv_n\,dx + \int_{\overline{\Omega}_n} R_n(\varphi_n u_n) : v_n\,d\mu_n \right) \\ &= \int_\Omega \frac{\tau}{(1+\tau|u|)^2|u|} \left(A\left(u \otimes u^t Du\right) : Dv - ADu : \left(v \otimes u^t Du\right)\right) dx \\ &+ \int_\Omega f \cdot \frac{v}{1+\tau|u|}\,dx + \int_\Omega GD\left[\frac{v}{1+\tau|u|}\right] dx. \end{aligned} \tag{5.75}$$

Proof. Since by linearity, the sequence τu_n satisfies (5.63) with f and G replaced by τf and τG, it is enough to show the result for $\tau = 1$.

Estimate (5.73) is a simple consequence of (5.70).

In order to show (5.75), we consider v_n and v which satisfy (5.62) and (5.74). Then, we have

$$\begin{aligned} &\int_{\Omega_n} AD(\varphi_n u_n) : Dv_n\,dx + \int_{\overline{\Omega}_n} R_n(\varphi_n u_n) : v_n\,d\mu_n \\ &= \int_{\Omega_n} \varphi_n ADu_n : Dv_n\,dx + \int_{\Omega_n} A(u_n \otimes \nabla\varphi_n) : Dv_n\,dx + \int_{\overline{\Omega}_n} R_n(\varphi_n u_n) : v_n\,d\mu_n. \end{aligned} \tag{5.76}$$

Taking into account that

$$\nabla\varphi_n = -\frac{1}{(1+\delta_n|u_n| + (1-\delta_n)|u|)^2} \left[\frac{\delta_n}{|u_n|}\,u_n^t Du_n + \frac{1-\delta_n}{|Pu|}\,(Pu)^t DPu \right], \tag{5.77}$$

we have

$$\begin{aligned} &\int_{\Omega_n} A(u_n \otimes \nabla\varphi_n) : Dv_n\,dx \\ &= -\int_{\Omega_n} \frac{\delta_n}{(1+\delta_n|u_n| + (1-\delta_n)|Pu|)^2|u_n|} A\left(u_n \otimes u_n^t Du_n\right) : Dv_n\,dx \\ &\quad - \int_{\Omega_n} \frac{1-\delta_n}{(1+\delta_n|u_n| + (1-\delta_n)|Pu|)^2|Pu|} A\left(u_n \otimes (Pu)^t DPu\right) : Dv_n\,dx. \end{aligned} \tag{5.78}$$

To estimate the first term on the right-hand side of ({refeqain2) we use that for every $m > 0$ we have

$$\begin{aligned} &\left| \int_{\Omega_n} \frac{\delta_n}{(1+\delta_n|u_n| + (1-\delta_n)|Pu|)^2|u_n|} A\left(u_n \otimes u_n^t Du_n\right) : Dv_n\,dx \right| \\ &\le Cm\delta_n \int_{\{|u_n|<m\}} |Du_n||Dv_n|\,dx + C\int_{\{|u_n|>m\}} |Du_n||Dv_n|\,dx. \end{aligned}$$

Therefore, passing to the limit first when n tends to infinity and then when m tends to infinity thanks to (5.70) and (5.74), we deduce

$$\lim_{n\to\infty}\int_{\Omega_n}\frac{\delta_n}{(1+\delta_n|u_n|+(1-\delta_n)|Pu|)^2|u_n|}A\left(u_n\otimes u_n^tDu_n\right):Dv_n\,dx=0. \tag{5.79}$$

For the second term in (5.78), we use the decomposition

$$\int_{\Omega_n}\frac{1-\delta_n}{(1+\delta_n|u_n|+(1-\delta_n)|Pu|)^2|Pu|}A\left(u_n\otimes (Pu)^tDPu\right):Dv_n\,dx$$
$$=\int_{\{|u_n-Pu|<1\}}\frac{1-\delta_n}{(1+\delta_n|u_n|+(1-\delta_n)|Pu|)^2|Pu|}A\left(u_n\otimes (Pu)^tDPu\right):Dv_n\,dx$$
$$+\int_{\{|u_n-Pu|\geq 1\}}\frac{1-\delta_n}{(1+\delta_n|u_n|+(1-\delta_n)|Pu|)^2|Pu|}A\left(u_n\otimes (Pu)^tDPu\right):Dv_n\,dx.$$

Taking into account that the measure of $\Omega_n\backslash\Omega$ tends to zero, that u_n converges pointwise to u a.e. in Ω, the weak convergence of v_n given in (5.62), and that a.e. on the set $\{|u_n-Pu|<1\}$, we have

$$\left|\frac{1-\delta_n}{(1+\delta_n|u_n|+(1-\delta_n)|Pu|)^2|Pu|}A\left(u_n\otimes (Pu)^tDPu\right)\right|\leq\frac{C|DPu|}{(1+\delta_n|u_n|+(1-\delta_n)|Pu|)},$$

we can use the Legesgue convergence dominte to prove that

$$\frac{1-\delta_n}{(1+\delta_n|u_n|+(1-\delta_n)|Pu|)^2|Pu|}A\left(u_n\otimes (Pu)^tDPu\right)\chi_{\{|u_n-Pu|<1\}}$$

converges strongly in $L^2(\tilde{\Omega})^{M\times N}$, which joining to the weak convergence of $Dv_n\chi_{\Omega_n}$ to $Dv\chi_\Omega$ in $L^2(\tilde{\Omega})^{M\times N}$, proves

$$\lim_{n\to\infty}\int_{\{|u_n-Pu|<1\}}\frac{1-\delta_n}{(1+\delta_n|u_n|+(1-\delta_n)|Pu|)^2|Pu|}A\left(u_n\otimes (Pu)^tDPu\right):Dv_n\,dx$$
$$=\int_\Omega\frac{1}{(1+|u|)^2|u|}A\left(u\otimes u^tDu\right):Dv\,dx.$$

By (5.71) we also have

$$\left|\int_{\{|u_n-Pu|\geq 1\}}\frac{1-\delta_n}{(1+\delta_n|u_n|+(1-\delta_n)|Pu|)^2|Pu|}A\left(u_n\otimes (Pu)^tDPu\right):Dv_n\,dx\right|$$
$$\leq\frac{C}{\delta_n}\left|\int_{\{|u_n-Pu|\geq 1\}}|DPu||Dv_n|dx\right|\to 0.$$

We have thus proved

$$\lim_{n\to\infty}\int_{\Omega_n}\frac{1-\delta_n}{(1+\delta_n|u_n|+(1-\delta_n)|Pu|)^2|Pu|}A\left(u_n\otimes(Pu)^tDPu\right):Dv_n\,dx$$
$$=\int_\Omega\frac{1}{(1+|u|)^2|u|}A\left(u\otimes u^tDu\right):Dv\,dx,$$

which joining to (5.78) and (5.79) permits to pass to the limit in the second term of (5.76) to deduce

$$\lim_{n\to\infty}\int_{\Omega_n}A(u_n\otimes\nabla\varphi_n):Dv_n\,dx=\int_\Omega\frac{1}{(1+|u|)^2|u|}A\left(u\otimes u^tDu\right):Dv\,dx. \tag{5.80}$$

In order to estimate the sum of the first and second terms in (5.76), we consider a number $\varepsilon>0$ and we write

$$\begin{aligned}
&\int_{\Omega_n}\varphi_nADu_n:Dv_n\,dx+\int_{\overline{\Omega}_n}R_n(\varphi_nu_n)\cdot v_n\,d\mu_n\\
&=\int_{\Omega_n}\varphi_nADu_n:D\left[\frac{\varepsilon|v_n|\,v_n}{1+\varepsilon|v_n|}\right]dx+\int_{\overline{\Omega}_n}R_n(\varphi_nu_n)\cdot\frac{\varepsilon|v_n|\,v_n}{1+\varepsilon|v_n|}\,d\mu_n\\
&+\int_{\Omega_n}ADu_n:D\left[\varphi_n\frac{v_n}{1+\varepsilon|v_n|}\right]dx+\int_{\overline{\Omega}_n}R_nu_n\cdot\left[\varphi_n\frac{v_n}{1+\varepsilon|v_n|}\right]d\mu_n\\
&-\int_{\Omega_n}ADu_n:\left[\frac{v_n}{1+\varepsilon|v_n|}\otimes\nabla\varphi_n\right]dx.
\end{aligned}\tag{5.81}$$

Let us estimate the right-hand side of this inequality. Thanks to (5.74) is immediate to show that

$$\lim_{\varepsilon\to0}\limsup_{n\to\infty}\left(\int_{\Omega_n}\varphi_nADu_n:D\left[\frac{\varepsilon|v_n|\,v_n}{1+\varepsilon|v_n|}\right]dx+\int_{\overline{\Omega}_n}R_n(\varphi_nu_n)\cdot\frac{\varepsilon|v_n|\,v_n}{1+\varepsilon|v_n|}\,d\mu_n\right)=0. \tag{5.82}$$

Assumption (5.63) shows

$$\begin{aligned}
&\lim_{n\to\infty}\left(\int_{\Omega_n}ADu_n:D\left[\varphi_n\frac{v_n}{1+\varepsilon|v_n|}\right]dx+\int_{\overline{\Omega}_n}R_nu_n\cdot\left[\varphi_n\frac{v_n}{1+\varepsilon|v_n|}\right]d\mu_n\right)\\
&=\int_\Omega f\cdot\frac{v}{(1+|u|)(1+\varepsilon|v|)}\,dx+\int_\Omega GD\left[\frac{v}{(1+|u|)(1+\varepsilon|v|)}\right]dx,
\end{aligned}\tag{5.83}$$

for every $\varepsilon>0$. Finally reasoning as to prove (5.80), we have

$$\begin{aligned}
&\lim_{n\to\infty}\int_{\Omega_n}ADu_n:\left[\frac{v_n}{1+\varepsilon|v_n|}\otimes\nabla\varphi_n\right]dx\\
&=\int_\Omega ADu:\left[\frac{v}{(1+\varepsilon|v|)(1+|u|)^2|u|}\otimes u^tDu\right]dx,
\end{aligned}\tag{5.84}$$

for every $\varepsilon > 0$.

So, passing to the limit in (5.81) first when n tends to infinity and then when ε tends to zero we deduce

$$\lim_{n\to\infty}\left(\int_{\Omega_n}\varphi_n ADu_n : Dv_n\,dx + \int_{\overline{\Omega}_n} R_n(\varphi_n u_n)\cdot v_n\,d\mu_n\right)$$
$$= \int_\Omega f\cdot\frac{v}{1+|u|}\,dx + \int_\Omega GD\left[\frac{v}{1+|u|}\right]dx - \int_\Omega ADu : \left[\frac{v}{(1+|u|)^2|u|}\otimes u^t Du\right]dx. \tag{5.85}$$

By (5.80) and (5.85) we can pass to the limit in (5.76) to deduce (5.75). □

Proof of Theorem 5.1. Let us divide the proof in five steps.

Step 1. We consider $f_n \in L^2(\Omega_n)^M$, $f \in L^2(\Omega)^M$, $G_n \in L^2(\Omega_n)^{M\times N}$, $G \in L^2(\Omega)^{M\times N}$, which satisfy (5.16), (5.17) and (5.18) and $v_n \in H^1(\Omega_n)^M$, $v \in H^1(\Omega)^M$ which satisfy (5.62) and (5.74). Let us prove that in this case

$$\lim_{n\to\infty}\left(\int_{\Omega_n} f_n\cdot v_n\,dx + \int_{\Omega_n} G_n : Dv_n\,dx\right) = \int_\Omega f\cdot v\,dx + \int_\Omega G : Dv\,dx. \tag{5.86}$$

For this purpose, we take $\rho > 0$ and we decompose

$$\int_{\Omega_n} f_n\cdot v_n\,dx + \int_{\Omega_n} G_n : Dv_n\,dx = \int_{\Omega_n\setminus\Omega^{\rho^-}} f_n\cdot v_n\,dx + \int_{\Omega_n\setminus\Omega^{\rho^-}} G_n : Dv_n\,dx$$
$$+ \int_{\Omega^{\rho^-}} f_n\cdot v_n\,dx + \int_{\Omega^{\rho^-}} G_n : Dv_n\,dx. \tag{5.87}$$

Using (5.18), the weak convergence of v_n to v in $H^1(\Omega^{\rho^-})^M$ and the Rellich-Kondrachov compactness theorem, we can pass to the limit in the two last terms of (5.87) to obtain

$$\lim_{n\to\infty}\left(\int_{\Omega^{\rho^-}} f_n\cdot v_n\,dx + \int_{\Omega^{\rho^-}} G_n : Dv_n\,dx\right) = \int_{\Omega^{\rho^-}} f\cdot v\,dx + \int_{\Omega^{\rho^-}} G : Dv\,dx.$$

By (5.16), (5.17), (5.18) and (5.11), we also have

$$\lim_{\rho\to 0}\limsup_{n\to\infty}\left|\int_{\Omega_n\setminus\Omega^{\rho^-}} f_n\cdot v_n\,dx + \int_{\Omega_n\setminus\Omega^{\rho^-}} G_n : Dv_n\,dx\right| = 0.$$

Therefore, passing to the limit in (5.87) first in n tends to infinty and then when ρ tends to zero, we get (5.86).

Step 2. We consider $\{h^m\}$, $\{H^k\}$ countable dense subsets of $L^2(\Omega)^M$ and $H^1(\Omega)^{M\times N}$ respectively. For every $m,k\in\mathbb{N}$, we denote by $w_n^{m,k}$ the solution of the variational problem

$$\begin{cases} w_n^{m,k}\in D_n \\ \displaystyle\int_{\Omega_n} ADw_n^{m,k}:Dv\,dx+\int_{\Omega_n}R_nw_n^{m,k}\cdot v\,d\mu_n=\int_{\Omega_n\cap\Omega}h^m\cdot v\,dx+\int_{\Omega_n\cap\Omega}H^m:Dv_n\,dx,\quad \forall\, v\in D_n,\end{cases}\tag{5.88}$$

Since $\|w_n^{m,k}\|_{H^1(\Omega_n)}$ is bounded independently of n, a diagonal argument provides $w^{m,k}\in H^1(\Omega)$ and a subsequence of n, still denoted by n such that

$$w_n^{m,k}\rightharpoonup w^{m,k}\ \text{ in } H^1(\Omega^{\rho^-}),\quad \forall\, m,k\in\mathbb{N}.\tag{5.89}$$

This will be the subsequence which appears in the statement of Theorem 5.1. Let us prove that for every $f_n\in L^2(\Omega_n)^M$, $G_n\in L^2(\Omega_n)^{M\times N}$, $f\in L^2_{loc}(\Omega)^M$ and $G\in L^2_{loc}(\Omega)^{M\times N}$, which satisfy (5.16), (5.17) and (5.18), there exists $u\in H^1(\Omega)^M$ such that the unique solution u_n of (5.19) converges weakly to u in $H^1(\Omega^{\rho^-})^M$, for every $\rho>0$ (without to extract any subsequence). To prove this result, it is enough to take sequences h^{m_j}, H^{k_j} which converge weakly in $L^2(\Omega)^M$ and $L^2(\Omega)^{M\times N}$ respectively to f and G respectively. Then, using $u_n-w_n^{m_j,k_j}$ as test functions in the differences of (5.19) and (5.89) (with $m=m_j$, $k=k_j$) and passing to the limit in n on the right-hand side thanks to (5.86), it is immediate to show that

$$\begin{aligned}&\limsup_{n\to\infty}\left(\|u_n-w_n^{m_j,k_j}\|^2_{H^1(\Omega_n)^M}+\int_{\Omega_n}R_n(u_n-w_n^{m_j,k_j}):(u_n-w_n^{m_j,k_j})d\mu_n\right)\\ &\le C\left(\|h^{m_j}-f\|^2_{L^2(\Omega)^M}+\|H^{m_j}-G\|^2_{L^2(\Omega)^{M\times N}}\right).\end{aligned}$$

This inequality implies that for every $\rho>0$, every cluster point u of u_n in the weak topology of $H^1(\Omega^{\rho^-})^M$ satisfies

$$\|u-w^{m_j,k_j}\|^2_{H^1(\Omega^{\rho^-})^M}\le C\left(\|h^{m_j}-f\|^2_{L^2(\Omega)^M}+\|H^{m_j}-G\|^2_{L^2(\Omega)^{M\times N}}\right),$$

which shows that u is the limit in $H^1_{loc}(\Omega)^M$ of w^{m_j,k_j}, and so, it is unique.

Step 3. For the subsequence of n constructed in Step 2, we define by D the space of functions in $H^1(\Omega)^M\cap L^\infty(\Omega)^M$ such that there exits a sequence $u_n\in\mathcal{D}_n$ satisfying (5.61) and (5.70) and such that there exist $f\in L^1(\Omega)^M$, $G\in L^2(\Omega)^{M\times N}$, satisfying that for every $v_n\in\mathcal{D}_n$, $v\in H^1(\Omega)^M\cap L^\infty(\Omega)^M$, which satisfies (5.62) and (5.74), we have (5.63).

We consider $u\in D$ and sequences and functions $u_n\in\mathcal{D}_n$, $f\in L^1(\Omega)^M$, $G\in L^2(\Omega)^{M\times N}$ such as it appears in the definition of the elements of D. For $v_n\in\mathcal{D}_n$, $v\in H^1(\Omega)^M\cap L^\infty(\Omega)^M$, which satisfies (5.62) and (5.74), we define $\lambda_n \in M(\mathbb{R}^N)$ by

$$\lambda_n(B) = \int_{\Omega_n\cap\Omega\cap B} A_n D(u_n-u) : D(v_n-v)\, dx + \int_{(\Omega_n\setminus\Omega)\cap B} A_n Du_n : Dv_n\, dx + \int_{\overline{\Omega}_n\cap B} R_n u_n \cdot v_n d\mu_n.$$

Let us prove that there exists $\lambda \in M(\mathbb{R}^N)$ such that

$$\lambda_n \overset{*}{\rightharpoonup} \lambda \ \text{ in } M(\mathbb{R}^N). \tag{5.90}$$

Moreover, the measure λ has support in $\overline{\Omega}$, it only depends on u and v (i.e. it does not depend on u_n, v_n, f and G) and for every $\varphi \in C_c^\infty(\Omega)$, it satisfies

$$\int_\Omega ADu : D(v\varphi)dx + \int_{\overline{\Omega}} \varphi\, d\lambda = \int_\Omega f \cdot v\varphi\, dx + \int_\Omega G : \cdot D(v\varphi)\, dx. \tag{5.91}$$

To prove this result, we take $\varphi \in C_c^\infty(\mathbb{R}^N)$. Applying (5.63) with v_n replaced by $v_n\varphi$, we have

$$\begin{aligned} &\exists \lim_{n\to\infty} \left(\int_{\Omega_n} ADu_n : Dv_n\, \varphi\, dx + \int_{\Omega_n} ADu_n : (v_n \otimes \nabla\varphi) dx + \int_{\overline{\Omega}_n} R_n u_n \cdot v_n \varphi\, d\mu_n \right) \\ &= \int_\Omega f \cdot v\varphi\, dx + \int_\Omega G : D(v\varphi)\, dx, \end{aligned} \tag{5.92}$$

but using (5.11) and the Rellich-Kondrachov's compactness theorem, it is simple to check that

$$\int_{\Omega_n} ADu_n : (v_n \otimes \nabla\varphi) dx \to \int_\Omega ADu : (v \otimes \nabla\varphi)\, dx$$

$$\int_{\Omega_n\cap\Omega} ADu : Dv\, \varphi) dx \to \int_\Omega ADu : Dv\, \varphi\, dx,$$

and therefore (5.92) can be written as

$$\begin{aligned} &\exists \lim_{n\to\infty} \left(\int_{\Omega_n\cap\Omega} AD(u_n - u) : Dv_n\, \varphi\, dx + \int_{\Omega_n\setminus\Omega} ADu_n : Dv_n\, \varphi\, dx + \int_{\overline{\Omega}_n} R_n u_n \cdot v_n \varphi\, d\mu_n \right) \\ &= \int_\Omega f \cdot v\, \varphi\, dx + \int_\Omega (G - ADu) : D(v\varphi)\, dx. \end{aligned}$$

which using also that

$$\int_{\Omega_n\cap\Omega} AD(u_n - u) : Dv\, \varphi\, dx \to 0,$$

finally gives

$$\exists \lim_{n\to\infty}\left(\int_{\Omega_n\cap\Omega} AD(u_n-u):D(v_n-v)\,\varphi\,dx+\int_{\Omega_n\setminus\Omega} ADu_n:Dv_n\,\varphi\,dx+\int_{\overline{\Omega}_n} R_n u_n\cdot v_n\varphi\,d\mu_n\right)$$
$$=\int_\Omega f\cdot v\varphi\,dx+\int_\Omega (G-ADu):D(v\varphi)\,dx,\quad \forall\,\varphi\in C_c^\infty(\mathbb{R}^N) \tag{5.93}$$

This proves that there exists the limit λ of λ_n in the sense of the distributions. Since λ_n is bounded in $M(\mathbb{R}^N)$ the limit holds in fact in the weak-$*$ sense of the measures and so, λ is a measure which by (5.93) has support in $\overline{\Omega}$ and satisfies (5.91). In order to prove that λ only depends on u and v, we consider other sequences and functions $\widetilde{u}_n$, $\widetilde{v}_n$, $\widetilde{f}$, $\widetilde{G}$ satisfying similar properties to u_n, v_n, f and G. By (5.63), we have

$$\int_{\Omega_n} AD(u_n-\widetilde{u}_n):D(u_n-\widetilde{u}_n)dx+\int_{\overline{\Omega}_n} R_n(u_n-\widetilde{u}_n)\cdot(u_n-\widetilde{u}_n)d\mu_n\to 0,$$

which immediately shows that

$$\left|\int_{\Omega_n\cap\Omega} AD(u_n-u):Dv_n\,\varphi\,dx+\int_{\Omega_n\setminus\Omega} ADu_n:Dv_n\,\varphi\,dx+\int_{\overline{\Omega}_n} R_n u_n\cdot v_n\varphi\,d\mu_n\right.$$
$$\left.-\int_{\Omega_n\cap\Omega} AD(\widetilde{u}_n-u):Dv_n\,\varphi\,dx+\int_{\Omega_n\setminus\Omega} AD\widetilde{u}_n:D\widetilde{v}_n\,\varphi\,dx+\int_{\overline{\Omega}_n} R_n\widetilde{u}_n\cdot\widetilde{v}_n\varphi\,d\mu_n\right|\to 0,$$

for every $\varphi\in C_c^\infty(\mathbb{R}^N)$. This proves that λ does not depend on u_n, v_n, f and G.

Step 4. For $u,v\in D$, we denote by $\nu(u,v)\in M(\overline{\Omega})$ the measure λ relative to u and v given in Step 3. Let us prove that the space D and the application $\nu: D\times D\to M(\mathbb{R}^N)$ defined in this way are in the conditions of Theorem 5.4.

It is very simple to check that ν is bilinear. In the following, for $u,v\in D$ we consider sequences and functions $u_n,v_n\in \mathcal{D}_n$, $f,\hat{f}\in L^1(\Omega)^M$, $G,\hat{G}\in L^2(\Omega)^{M\times N}$ in the conditions which appear in the definition of the elements of D, relative to u and v respectively.

In order to prove (5.27), we consider $\varphi\in C^1(\mathbb{R}^N)$, then, for every $z_n\in D_n$, $z\in H^1(\Omega)^m\cap L^\infty(\Omega)^M$ satisfying (5.62) and (5.74), assumption (5.11), the Rellich-Kondrachov's compactness theorem and (5.63) give

$$\begin{aligned}
&\int_{\Omega_n} AD(u_n\varphi) : Dv_n\,dx + \int_{\overline{\Omega}} R_n(u_n\varphi)\cdot v_n\,d\mu_n \\
&= \int_{\Omega_n} ADu_n : D(v_n\varphi)\,dx + \int_{\overline{\Omega}} R_n(u_n\varphi)\cdot v_n\,d\mu_n \\
&\quad + \int_{\Omega_n} A(u_n\otimes\nabla\varphi) : Dv_n\,dx - \int_{\Omega_n} ADu_n : (v_n\otimes\nabla\varphi)\,dx \\
&\to \int_\Omega f\cdot(v\varphi)dx + \int_\Omega G : D(v\varphi)\,dx + \int_\Omega A(u\otimes\nabla\varphi) : Dv\,dx - \int_\Omega ADu : (v\otimes\nabla\varphi)\,dx.
\end{aligned}$$

This proves that $u\varphi$ belongs to D and that $\nu(u\varphi, v)$ is the limit in the weak-$*$ sense of the measures of ζ_n defined as

$$\begin{aligned}
\zeta_n(B) &= \int_{\Omega_n\cap\Omega\cap B} A_n D\left[(u_n-u)\varphi\right] : D(v_n-v)\,dx \\
&\quad + \int_{(\Omega_n\setminus\Omega)\cap B} A_n D(u_n\varphi) : Dv_n\,dx + \int_{\overline{\Omega}_n\cap B} R_n(u_n\varphi)\cdot v_n d\mu_n \\
&= \int_{\Omega_n\cap\Omega\cap B} A_n D(u_n-u) : D(v_n-v)\,\varphi\,dx + \int_{\Omega_n\cap\Omega\cap B} A_n((u_n-u)\otimes\nabla\varphi) : D(v_n-v)\,dx \\
&\quad + \int_{(\Omega_n\setminus\Omega)\cap B} A_n Du_n : Dv_n\,\varphi\,dx + \int_{(\Omega_n\setminus\Omega)\cap B} A_n(u_n\otimes\varphi) : Dv_n\,dx + \int_{\overline{\Omega}_n\cap B} R_n(u_n\varphi)\cdot v_n d\mu_n,
\end{aligned}$$

which is simple to check that agrees with $\nu(u,v)\varphi$. Similarly, we can show that $\nu(u, v\varphi) = \nu(u,v)\varphi$. This proves (5.27).

Inequality (5.28) is immediate from the definition of ν, A in $L^\infty(\widetilde{\Omega};\mathcal{T}_{M\times N})$, and the second assertion in assumption (5.9).

Now, consider $u\in D$ and a sequence $u^m\in D$ which converges weakly to u in $H^1(\Omega)^M$. For $m\in\mathbb{N}$, taking into account Remark 5.7 and Step 1, we have that $u^m/(1+\varepsilon|u^m|)$ belongs to D, for every $\varepsilon>0$. Moreover, taking a sequence $u_n^m\in D_n$ associated to u^m such that it appears in the definition of the elements of D, and δ_n converging to zero and such that (5.71) holds, we have that the sequence $u_n^{m,\varepsilon} = \psi_n^{m,\varepsilon}u_n^m$, with

$$\psi_n^{m,\varepsilon} = \frac{1}{1+\delta_n|u_n^m| + (\varepsilon-\delta_n)|Pu^m|},$$

(Pu^m a prolongation of u^m to a function in $H_0^1(\widetilde{\Omega})^M$) belongs to D_n and satisfies

$$\limsup_{n\to\infty}\left(\int_{\Omega_n\cap\Omega} A_n Du_n^{m,\varepsilon} : Du_n^{m,\varepsilon}\,dx + \int_{\overline{\Omega}_n} R_n u_n^{m,\varepsilon}\cdot u_n^{m,\varepsilon}\,d\mu_n\right) < +\infty.$$

Then, for every $\varphi \in C^1(\overline{\Omega})$, $\varphi \geq 0$ in Ω, statement (5.63) and

$$\int_{\Omega_n \cap \Omega} ADu : D\left[(u_n - u_n^{m,\varepsilon})\varphi\right] \, dx \to \int_\Omega ADu : D\left[(u_n - u_n^{m,\varepsilon})\varphi\right] \, dx$$

prove

$$\begin{aligned} &\int_{\Omega_n} AD(u_n - u) : D\left[(u_n - u_n^{m,\varepsilon})\varphi\right] \, dx + \int_{\overline{\Omega}_n} R_n u_n \cdot (u_n - u_n^{m,\varepsilon})\varphi \, d\mu_n \\ &\to \int_\Omega f \cdot \left(u - \frac{u^m}{1+\varepsilon|u^m|}\right)\varphi dx + \int_\Omega (G - ADu) : D\left[\left(u - \frac{u^m}{1+\varepsilon|u^m|}\right)\varphi\right] dx, \end{aligned} \tag{5.94}$$

By (5.11) and the Rellich-Kondrachov compactness theorem, we have

$$\int_{\Omega_n} ADu_n : \left((u_n - u_n^{m,\varepsilon}) \otimes \nabla\varphi\right) \, dx \to \int_\Omega ADu : \left(\left(u - \frac{u^m}{1+\varepsilon|u^m|}\right) \otimes \nabla\varphi\right) \, dx.$$

Moreover, the choice of δ_n easily implies

$$\int_{\Omega_n} \left| Du_n^{m,\varepsilon} - \frac{Du_n^m}{1+\varepsilon|Pum|} - Pu^m \otimes \nabla\left[\frac{1}{1+\varepsilon|Pu^m|}\right]\right|^2 dx \to 0.$$

Thus, we easily deduce from (5.94)

$$\begin{aligned} &\int_{\Omega_n \cap \Omega} AD(u_n - u) : \left(Du_n - \frac{Du_n^m}{1+\varepsilon|Pu^m|}\right)\varphi \, dx \\ &+ \int_{\Omega_n \setminus \Omega} ADu_n : \left(Du_n - \frac{Du_n^m}{1+\varepsilon|Pu^m|}\right)\varphi \, dx + \int_{\overline{\Omega}_n} R_n u_n \cdot (u_n - u_n^{m,\varepsilon})\varphi \, d\mu_n \\ &\to \int_\Omega f \cdot \left(u - \frac{u^m}{1+\varepsilon|u^m|}\right)\varphi dx + \int_\Omega (G - ADu) : D\left[\left(u - \frac{u^m}{1+\varepsilon|u^m|}\right)\varphi\right] dx, \end{aligned} \tag{5.95}$$

which using that

$$\int_{\Omega_n \cap \Omega} AD(u_n - u) : \left(Du - \frac{DPu^m}{1+\varepsilon|Pu^m|}\right) \varphi \, dx \to 0,$$

can be written as

$$\begin{aligned} &\int_{\Omega_n \cap \Omega} AD(u_n - u) : \left(D(u_n - u) - \frac{D(u_n^m - u^m)}{1+\varepsilon|u^m|}\right) \varphi \, dx \\ &+ \int_{\Omega_n \setminus \Omega} ADu_n : \left(Du_n - \frac{Du_n^m}{1+\varepsilon|Pu^m|}\right) \varphi \, dx + \int_{\overline{\Omega}_n} R_n u_n \cdot (u_n - u_n^{m,\varepsilon})\varphi \, d\mu_n \\ &\to \int_\Omega f \cdot \left(u - \frac{u^m}{1+\varepsilon|u^m|}\right)\varphi dx + \int_\Omega (G - ADu) : D\left[\left(u - \frac{u^m}{1+\varepsilon|u^m|}\right)\varphi\right] dx, \end{aligned}$$

Using here Young's inequality, the second assertion in (5.9) and the definition of ν, we have

$$\int_{\overline{\Omega}} \varphi \, d\nu(u,u) = \lim_{n\to\infty}\Bigg(\int_{\Omega_n\cap\Omega} AD(u_n-u):D(u_n-u)\,\varphi\,dx$$

$$+\int_{\Omega_n\setminus\Omega} ADu_n:Du_n\varphi\,dx+\int_{\overline{\Omega}_n} R_nu_n\cdot u_n\varphi\,d\mu_n\Bigg)$$

$$\leq \gamma\liminf_{n\to\infty}\Bigg(\int_{\Omega_n\cap\Omega}\frac{AD(u_n^m-u^m):D(u_n^m-u^m)}{(1+\varepsilon|u^m|)^2}\varphi\,dx$$

$$+\int_{\Omega_n\setminus\Omega}\frac{ADu_n^m:Du_n^m}{(1+\varepsilon|u^m|)^2}\varphi\,dx+\int_{\overline{\Omega}_n}\frac{R_nu_n^m\cdot u_n^m}{(1+\varepsilon|u^m|)^2}\varphi\,d\mu_n$$

$$+\int_\Omega f\cdot\left(u-\frac{u^m}{1+\varepsilon|u^m|}\right)\varphi dx+\int_\Omega(G-ADu):D\left[\left(u-\frac{u^m}{1+\varepsilon|u^m|}\right)\varphi\right]dx$$

$$\leq\gamma\int_{\overline{\Omega}}\varphi\,d\nu(u^m,u^m)+\int_\Omega f\cdot\left(u-\frac{u^m}{1+\varepsilon|u^m|}\right)\varphi dx+\int_\Omega(G-ADu):D\left[\left(u-\frac{u^m}{1+\varepsilon|u^m|}\right)\varphi\right]dx,$$

for a constant γ which only depends on α, $\|A\|_{L^\infty(\Omega;\mathcal{L}(\mathcal{M}_N,\mathcal{M}_N))}$ and β. Taking the limit in this inequality when ε tends to zero and then the liminf in n we conclude

$$\int_{\overline{\Omega}}\varphi\,d\nu(u,u)\leq\gamma\liminf_{m\to\infty}\int_{\overline{\Omega}}\varphi\,d\nu(u^m,u^m),$$

which proves that ν also satisfies (5.29).

Step 5. Using Theorem 5.4, we know there exists a measure $\mu\in M(\overline{\Omega})$, which vanish on the set of capacity zero, a μ-measurable function $R:\overline{\Omega}\to\mathcal{M}_{M\times N}$ and an application $V:\overline{\Omega}\to\mathcal{V}_M$ such that defining the space $\overline{D}$ by (5.32) endowed with the norm defined by (5.33), we have that D is dense in $\overline{D}$ and that (5.34) holds. Let us prove that μ, R and V are in the conditions of the thesis of Theorem 5.4.

We consider $f_n\in L^2(\Omega_n)^M$, $G_n\in L^2(\Omega_n)^{M\times N}$, $f\in L^2_{loc}(\Omega)^M$ and $G\in L^2_{loc}(\Omega)^{M\times N}$, which satisfy (5.16), (5.17) and (5.18). Taking u_n as the solution of problem (5.19), we can apply Proposition 5.8 and (5.91) with $\varphi=1$ to deduce that for every $\tau>0$, the function $u/(1+\tau|u|)$ is in D and satisfies

$$\int_\Omega AD\left[\frac{u}{1+\tau|u|}\right]:Dv\,dx+\int_{\overline{\Omega}}R\left[\frac{u}{1+\tau|u|}\right]\cdot v\,d\mu$$
$$=\int_\Omega\frac{\tau}{(1+\tau|u|)^2|u|}\left(A\left(u\otimes u^tDu\right):Dv-ADu:\left(v\otimes u^tDu\right)\right)dx \tag{5.96}$$
$$+\int_\Omega f\cdot\frac{v}{1+\tau|u|}\,dx+\int_\Omega GD\left[\frac{v}{1+\tau|u|}\right]dx,$$

for every $v \in D$. Using in this equality $v = u/(1+\tau|u|)$ we deduce that

$$\limsup_{\tau\to 0}\left(\left\|\frac{u}{1+\tau|u|}\right\|^2_{H^1(\Omega)^M} + \int_{\overline{\Omega}} R\left[\frac{u}{1+\tau|u|}\right]\cdot\left[\frac{u}{1+\tau|u|}\right]d\mu\right) < +\infty.$$

Therefore, since $u/(1+\tau|u|)$ converges to u in $H^1(\Omega)^M$ when τ tends to zero and then in capacity, we can apply the Vitali theorem to obtain that $Ru\cdot u$ belongs to $L^1(\overline{\Omega})$. This proves that u belongs to $\overline{D}$. Moreover, passing to the limit when τ tends to zero in (5.96), we deduce that u satisfies

$$\int_\Omega ADu : Dv\,dx + \int_{\overline{\Omega}} Ru\cdot v\,d\mu = \int_\Omega f\cdot v\,dx + \int_\Omega GDv\,dx,$$

for every $v \in D$ and then by density for every $v \in \overline{D}$. This proves that u is the unique solution of (5.20). $\square$

Let us now give the proof of Theorem 5.2. first, we need to show that Korn's constant for Ω_n is bounded. This will be a consequence of the following results

Lemma 5.9 *For every bounded open set $O \subset \mathbb{R}^{N-1}$, which is star-shaped with respect to every point of a ball, there exists a constant $C > 0$ such that for every $\phi \in W^{1,\infty}(O)$, with $\|\nabla\phi\|_{L^\infty(O)^{N-1}} < 1/(8\,diam(O))$, $1/2 < \phi < 3/2$ in O, we have*

$$\left\|p - \frac{1}{|\vartheta_\phi|}\int_{\vartheta_\phi} p\,dx\right\|_{L^2(\vartheta_\phi)} \le C\|\nabla p\|_{H^{-1}(\vartheta_\phi)^N}, \quad \forall\, p \in L^2(\vartheta_\phi), \tag{5.97}$$

where we have denoted

$$\vartheta_\phi = \{(x', x_N) \in \mathbb{R}^{N-1}\times\mathbb{R} : \quad x' \in O,\ 0 < x_n < \phi(x')\}.$$

Proof. Using a translation, we can assume that O is star-shaped with respect to a ball $B(0,\rho) \subset O$. Then, taking $\varepsilon \in (0, \min\{\rho, 1/8\})$ and ϕ in the assumptions of the Lemma, let us show

$$\vartheta_\phi \text{ is star-shaped with respect to every point of } B((0,1/4),\varepsilon) \subset \mathbb{R}^{N-1}\times\mathbb{R}, \tag{5.98}$$

which by Lemma 3.1 in Chapter 3 of [21] will prove the result.

Let us argue by contradiction. If (5.98) is not true, then there exist $(y', y_N) \in B((0,1/4),\varepsilon)$ and two different points (x', x_N), $(\hat{x}', \hat{x}_N) \in \partial\vartheta_\phi$ such that

$$(x' - y', x_N - y_N) = \lambda(\hat{x}' - x', \hat{x}_N - x_N) \ \text{ with } \lambda > 0 \tag{5.99}$$

If x' belongs to ∂O, then, since O is star-shaped with respect to $B(0,\rho)$, (5.99) implies that $\hat{x}'=x'$, which gives the contradiction $\lambda=0$.

If x' does not belongs to O, then $x_N=\phi(x')$, which by $\hat{x}_N\leq\phi(\hat{x}')$, (5.99), $y_N<1/4+\varepsilon$ and $\phi(x')>1/2$ gives

$$\frac{\phi(\hat{x}')-\phi(x')}{|\hat{x}'-x'|}\geq\frac{\hat{x}_N-\phi(x')}{|\hat{x}'-x'|}=\frac{\hat{x}_N-x_N}{|\hat{x}'-x'|}=\frac{x_N-y_N}{|x'-y'|}=\frac{\phi(x')-y_N}{|x'-y'|}\geq\frac{1/4-\varepsilon}{diam(O)}>\frac{1}{8\,diam(O)},$$

in contradiction with $\|\nabla\phi\|_{L^\infty(O)^{N-1}}<1/(8\,diam(O))$.

□

Theorem 5.10 *We consider a bounded connected open set $\omega\subset\mathbb{R}^{N-1}$ satisfying the uniform exterior cone condition and a constant $M>0$. Then, there exists a constant $C>0$ such that denoting*

$$\Theta=\{(x',x_N)\in\mathbb{R}^{N-1}\times\mathbb{R}:\quad x'\in\omega,\ 0<\phi(x')<x_N\},\tag{5.100}$$

with $\phi\in W^{1,\infty}(\omega)$, $1/2<\phi<3/2$, $\|\nabla\phi\|_{L^\infty(\omega)^{N-1}}\leq M$, we have

$$\left\|p-\frac{1}{|\Theta|}\int_\Theta p\,dx\right\|_{L^2(\Theta)}\leq C\|\nabla p\|_{H^{-1}(\Theta)^N},\quad\forall\,p\in L^2(\Theta).\tag{5.101}$$

Proof. Since ω satisfies the uniform cone condition, there exist open sets $O_1,\cdots,O_k$, which are star-shaped with respect to every point of a ball, they have diameter less than $1/(8M)$ and satisfy $\omega=\cup_{i=1}^k O_i$. Using then that $\Theta=\cup_{i=1}^k\vartheta_i$ with

$$\vartheta_i=\{(x',x_N)\in\mathbb{R}^{N-1}\times\mathbb{R}:\quad x'\in O_i,\ 0<x_n<\phi(x')\},\quad 1\leq i\leq k.$$

Lemma 5.9 and Theorem 3.1 in Chapter 3 of [21] give the result. □

As a consequence, we have

Theorem 5.11 *We consider a bounded connected open set $\omega\subset\mathbb{R}^{N-1}$ satisfying the uniform exterior cone condition and a constant $M>0$. Then, there exists a constant $C>0$ such that defining Θ by (5.100), with ϕ as in Theorem 5.10, we have*

$$\|u\|_{H^1(\Theta)^N}\leq C\|e(u)\|_{L^2(\Theta)^{N\times N}},\quad\forall\,u\in H^1(\Theta)^N,\quad u=0\ \text{ on }\omega\times\{0\}.\tag{5.102}$$

Proof. Thanks to the equality

$$\partial^2_{jk}u^i=\partial_k e_{ij}(u)+\partial_j e_{ik}(u)-\partial_i e_{jk}(u),\quad\forall\,u\in H^1(\Theta)^N\quad\forall\,i,j,k\in\{1,\cdots,N\},$$

we can apply (5.100) to $\partial_j u^i$, $1 \leq i,j \leq N$ to deduce

$$\left\|\partial_j u^i - \frac{1}{|\Theta|}\int_\Theta \partial_j u^i\,dz\right\|_{L^2(\Theta)} \leq C\|e(u)\|_{L^2(\Theta)^{N\times N}}, \quad \forall\, u \in H^1(\vartheta)^N, \tag{5.103}$$

where C only depends on M and ω. Applying also (5.100) to

$$p = u^i - \frac{1}{|\Theta|}\int_\Theta \nabla u^i\,dz\cdot x, \quad i \in \{1,\cdots,N\},$$

and taking into account (5.103), we have

$$\begin{gathered}\left\|u^i - \frac{1}{|\Theta|}\int_\Theta u^i\,dz - \frac{1}{|\Theta|}\int_\Theta \nabla u^i\,dz\cdot(x-x_0)\right\|_{L^2(\Theta)} \leq C\left\|\nabla u^i - \frac{1}{|\Theta|}\int_\Theta \nabla u^i\,dz\right\|_{H^{-1}(\Theta)^N}\\ \leq C\left\|\nabla u^i - \frac{1}{|\Theta|}\int_\Theta \nabla u^i\,dz\right\|_{L^2(\Theta)^N} \leq C\|e(u)\|_{L^2(\Theta)^{N\times N}}, \quad i \in \{1,\cdots,N\}\end{gathered} \tag{5.104}$$

where x_0 denotes the center of mass of Θ. Let us prove that these inequalities imply (5.102) (for another constant C). We argue by contradiction: If (5.102) is not true, then there exists $\phi_n \in W^{1,\infty}(\omega)$, $1/2 < \phi_n < 3/2$, $\|\nabla\phi_n\|_{L^\infty(\omega)^{N-1}} \leq M$, and $u_n \in H^1(\Theta_n)^N$, with Θ_n defined by (5.100) for $\phi = \phi_n$, such that

$$u_n = 0 \text{ on } \omega\times\{0\}, \quad \|u_n\|_{H^1(\Theta_n)^N} = 1, \tag{5.105}$$

$$\|e(u_n)\|_{L^2(\Theta_n)^{N\times N}} \leq \frac{1}{n}. \tag{5.106}$$

Since ϕ_n is bounded in $W^{1,\infty}(\omega)$, we can assume that there exists $\phi \in W^{1,\infty}(\omega)$ such that ϕ_n converges uniformly to ϕ in $\overline{\omega}$. Defining Θ and Θ^ρ, $\rho > 0$, by (5.100) with $\phi = \phi$ and $\phi = \phi - \rho$ respectively and using (5.105), we conclude the existence of $u \in H^1(\Theta)^N$, with $u = 0$ on $\omega\times\{0\}$ such that

$$u_n \rightharpoonup u \ \text{ in } H^1(\Theta^\rho)^N,\ \forall\,\rho > 0, \tag{5.107}$$

which joining to (5.105) and $|(\Theta_n\setminus\Theta)\cup(\Theta\setminus\Theta_n)|$ converging to zero implies that

$$\frac{1}{|\Theta_n|}\int_{\Theta_n} u_n\,dz \to \frac{1}{|\Theta|}\int_\Theta u\,dz, \quad \frac{1}{|\Theta_n|}\int_{\Theta_n} Du_n\,dz \to \frac{1}{|\Theta|}\int_\Theta Du\,dz,$$

and that the center of mass x_0^n of Θ_n converges to the of mass x_0 of Θ. Therefore, (5.106) and inequalities (5.103), (5.104) applied in Θ_n, prove

$$\left\|u_n - \frac{1}{|\Theta|}\int_\Theta u\,dz - \frac{1}{|\Theta|}\int_\Theta Du\,dz\,(x-x_0)\right\|_{H^1(\Theta_n)^N} \to 0. \tag{5.108}$$

In particular, this implies that u is linear and since $u = 0$ on $\omega \times \{0\}$ we get that u is the zero function. By (5.108), this proves that $\|u_n\|_{H^1(\Theta_n)^N}$ converges to zero in contradiction with (5.105). $\square$

By Theorem 5.11, we can now prove

Proof of Theorem 5.2. By the uniform convergence to zero of the functions ψ_n, is clear that (5.8) is satisfied. So, applying Theorem 5.1, with $\mu_n = 0$, and

$$A\xi = B\left(\frac{\xi + \xi^t}{2}\right),$$

it is enough to show that there exist $\alpha > 0$ and a sequence ρ_n converging to zero such that

$$\alpha\|u\|^2_{H^1(\Omega_n)^N} \leq \int_{\Omega_n} Ae(u) : e(u)\, dx, \quad \forall\, u \in H^1(\Omega_n)^N,\ u \in V_n(x) \text{ q.e. in } \overline{\Omega}_n \tag{5.109}$$

$$\|u\|^2_{L^2(\Omega_n\setminus\Omega)} \leq \rho_n \int_{\Omega_n} Ae(u) : e(u)\, dx, \quad \forall\, u \in H^1(\Omega_n)^N,\ u \in V_n(x) \text{ q.e. in } \overline{\Omega}_n. \tag{5.110}$$

Inequality (5.109) is a simple consequence of (5.22), (5.102) and $V_n(x) = \{0\}$ for every $x \in \omega \times \{0\}$.

In order to show (5.110), we use again that $V_n = \{0\}$ in $\omega \times \{0\}$, which gives that

$$|u(x', x_N)|^2 = \left(\int_0^{x_N} \partial_N u(x', t)\, dt\right)^2 \leq (1 + \|\psi_n\|_{L^\infty(\omega)}) \int_0^{1+\psi_n(x')} |\partial_N u(x', t)|^2 dt,$$

a.e. $x \in \Omega_n$, for every $u \in H^1(\Omega_n)^N$, with $u \in V_n(x)$ q.e. in $\overline{\Omega}_n$. Integrating in $\Omega_n \cap \{x_N > 1 - \varepsilon\}$ with $\varepsilon > 0$, we then get

$$\int_{\Omega_n \cap \{x_N > 1-\varepsilon\}} |u(x', x_N)|^2 dx \leq C\left(\|\psi_n\|_{L^\infty(\omega)} + \varepsilon\right) \int_{\Omega_n} |\partial_N u|^2 dx,$$

for every $u \in H^1(\Omega_n)^N$, $u \in V_n(x)$ q.e. in $\overline{\Omega}_n$. Thanks to (5.109) this proves (5.110). $\square$

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